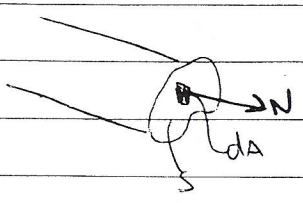
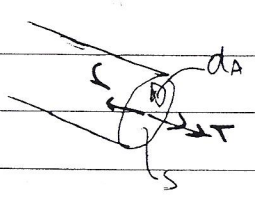


P3

Propriedades das Figuras Planas



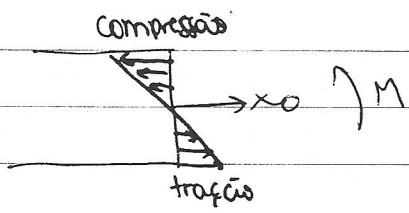
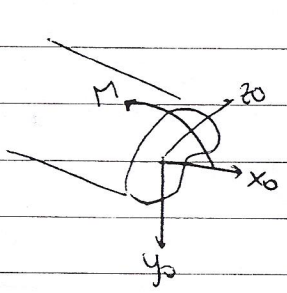
$$\bar{N} = \frac{N}{A}, \quad A = \int_S dA \text{ (cm}^2\text{)}$$



$$\bar{T} = \frac{T}{A}, \quad I_0 = \int_S r^2 dA \text{ [m}^4\text{]}$$

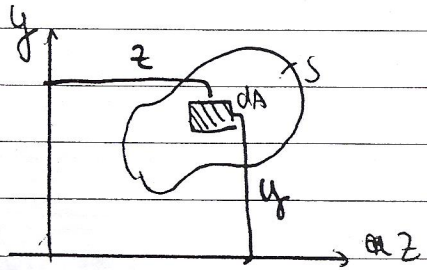
momento polar de inércia

Ainda vamos ver...



$$\sigma = \frac{M}{I_{z0}} y$$

momento de inércia da seção transversal em relação ao eixo z_0



Momento estático de S

$$M_{sy} = \int_S z dA, \text{ em relação ao eixo } y$$

Momento de inércia de S

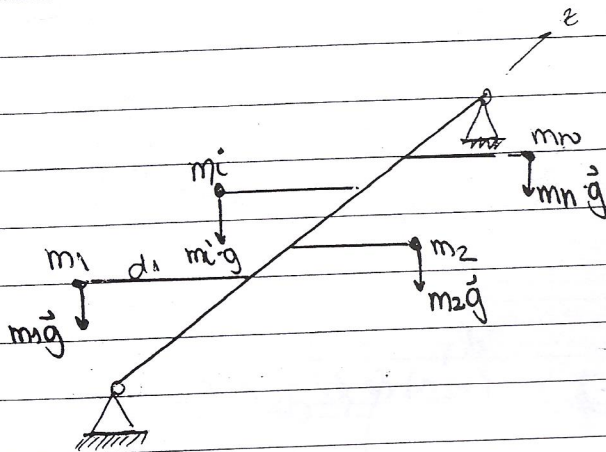
$$I_{sz} = \int_S y^2 dA, \text{ em relação ao eixo } z$$

$$I_{yz} = \int_S zy dA, \text{ em relação aos eixos } y \text{ e } z$$

$$I_z = \int_S y^2 dA, \text{ em relação ao eixo } z$$

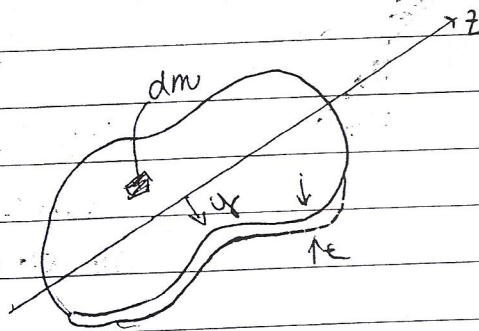


Propriedades do momento estático



eq. em torno do eixo z

$$\sum m_i d_i = 0$$



- espessura t constante
- densidade ρ constante

$$dm = t \cdot \rho \cdot dA$$

$$\int y dm = 0$$

para o eixo z

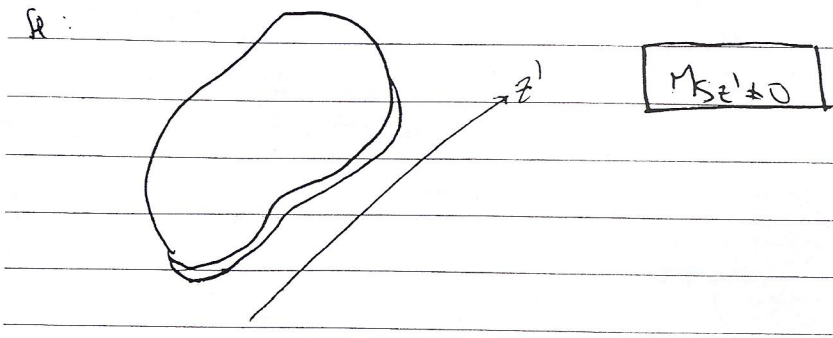
$$\int y (t \rho dA) = 0$$

$$t \rho \int y dA = 0$$

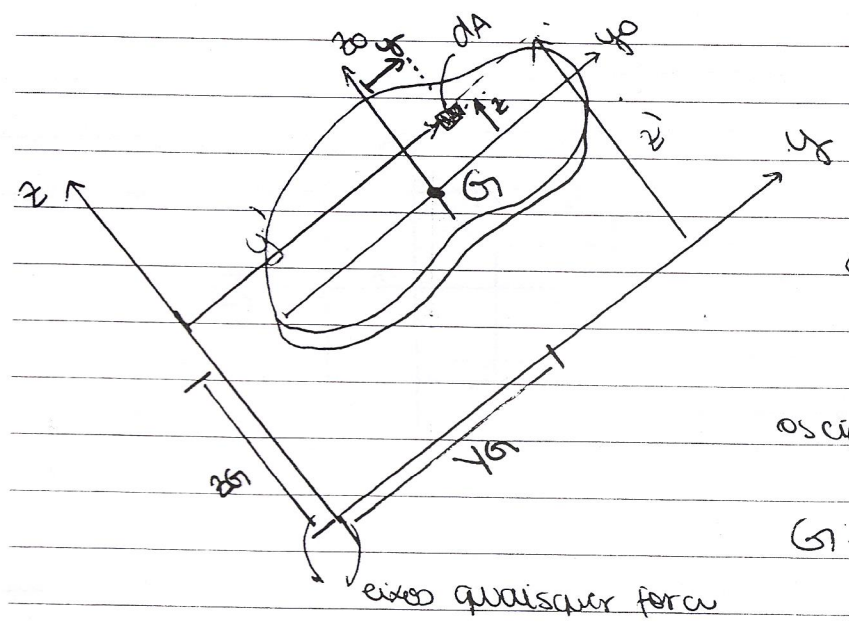
$$\Rightarrow \int y dA = 0$$

logo, p/ equilíbrio: $M_{sz} = 0$





• Barricentro (ou centro de massa) de uma figura plana



y, z : distância em rel. aos eixos z_0, y_0 até dA

y', z' : distância em rel. aos eixos y, z até dA

y_G, z_G : distância entre os eixos y, z e y_0, z_0 .

G : barricentro

eixo quaisquer força

$$y' = y + y_G$$

$$z' = z + z_G$$

$$M_{S z_0} = \int y dA = 0$$

$$M_{S z'} = \int y' dA = \int (y + y_G) dA$$

$$M_{S z'} = \int y dA + \int y_G dA = M_{S z_0} + y_G \int dA$$

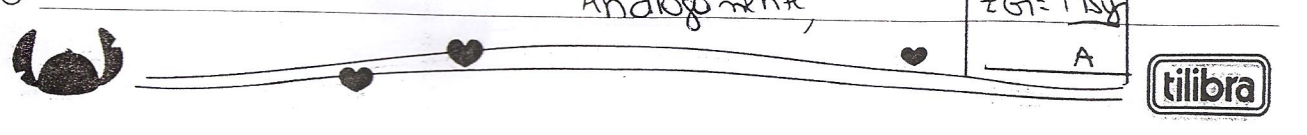
$$M_{S y_0} = \int z dA = 0$$

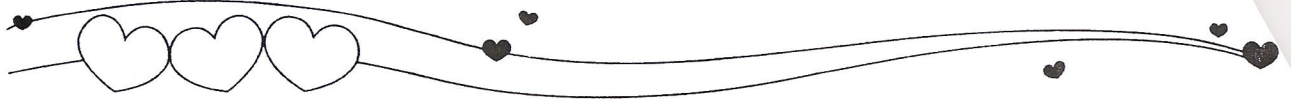
$$M_{S z'} = y_G A \Rightarrow y_G = \frac{M_{S z'}}{A}$$

$y_G = \frac{M_{S z'}}{A}$
$z_G = \frac{M_{S y'}}{A}$

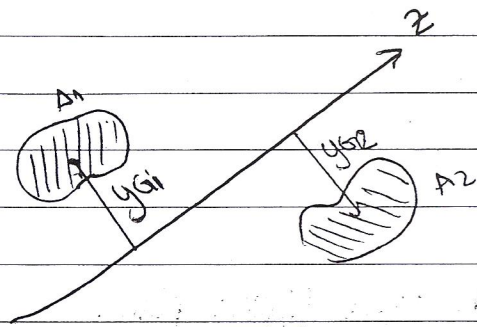
Analogamente,

© Disney



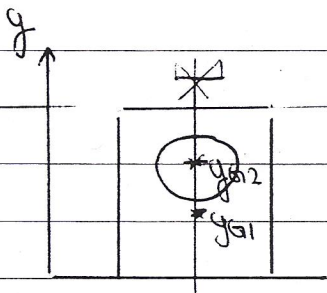
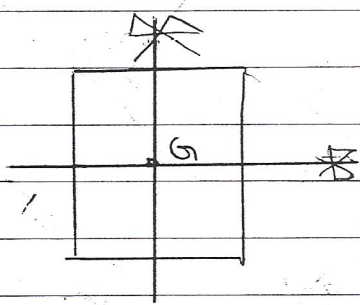
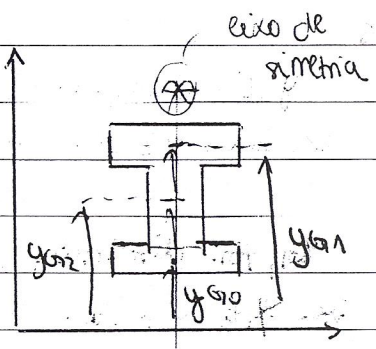


$$M_{SZ}^{TOTAL} = \sum_{i=1}^n y_{Gi} \cdot A_i = y_G \cdot A_{TOTAL}$$



$$y_G = \frac{\sum y_{Gi} \cdot A_i}{\sum A_i}$$

média ponderada das distâncias dos centros de massa ponderado pelas áreas

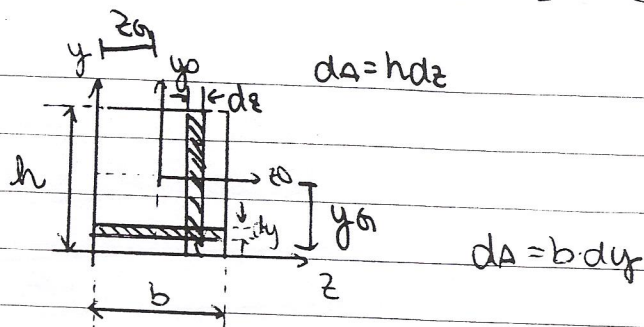


$$y_G = \frac{y_{G1} A_1 - y_{G2} A_2}{A_1 - A_2}$$



Exemplo

quanto vale y_G, z_G :



$$M_{S_z} = \int_S y \, dA = \int_0^h y (b \, dy)$$

$$M_{S_z} = b \int_0^h y \, dy = b \left[\frac{y^2}{2} \right]_0^h = \frac{bh^2}{2}$$

$$\Rightarrow y_G = \frac{M_{S_z}}{A} = \frac{bh^2/2}{bh} = \frac{h}{2}$$

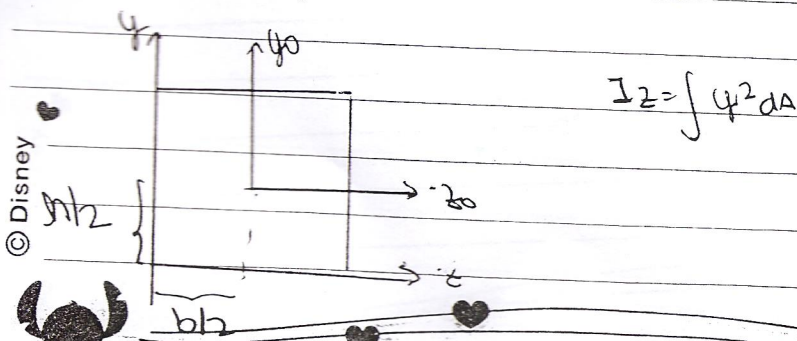
$$M_{S_y} = \int_S z \, dA = \int_0^b z (h \, dz)$$

$$M_{S_y} = h \int_0^b z \, dz = h \left[\frac{z^2}{2} \right]_0^b = \frac{hb^2}{2}$$

$$\Rightarrow z_G = \frac{M_{S_y}}{A} = \frac{hb^2/2}{hb} = \frac{b}{2}$$

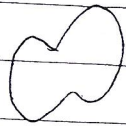
Exercício: Fazer em casa

$I_z, I_y, I_{z_G}, I_{y_G} \Rightarrow$



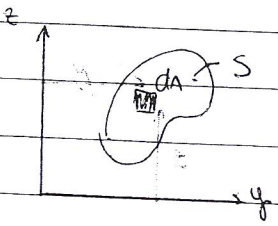


Exercícios (mini-revisão da aula passada)



$$\begin{cases} A = \int dA \\ J_0, I_{20} \end{cases}$$

$$\bar{v} = N/A, \quad \bar{c} = \bar{r} / J_0, \quad \bar{v} = M / I_{20} g$$



Momentos de inércia de S:

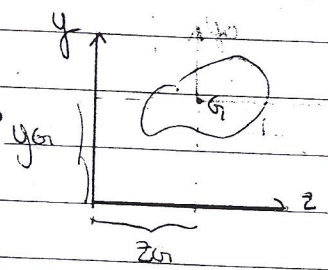
$$M_{sy} = \int_S z dA, \quad M_{sz} = \int_S y dA$$

Momentos de inércia de S:

$$I_z = \int_S y^2 dA, \quad I_y = \int_S z^2 dA$$

No baricentro: $M_s = 0$

⇒ Pl um eixo qualquer:

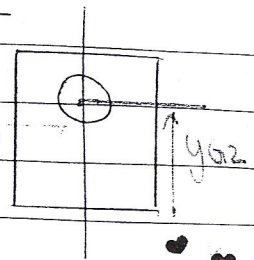
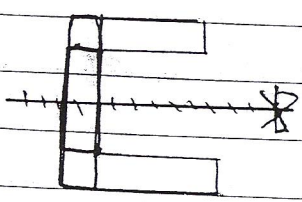


$$y_G = \frac{M_{sz}}{A}$$

$$M_{sz0} = M_{sy0} = 0$$

$$z_G = \frac{M_{sy}}{A}$$

* eixos principais são eixos de simetria e houver tais eixos

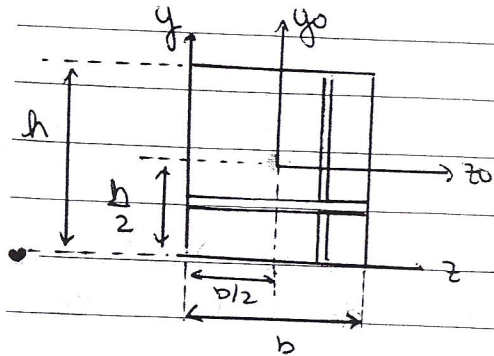


$$y_G = \frac{y_{G1} \cdot A_1 - y_{G2} \cdot A_2}{A_1 - A_2}$$



matéria: ...

(exercício da aula passada)



$$A = bh$$
$$dA = hdy$$
$$dA = dydz$$

$$I_z = \frac{bh^3}{3}$$

$$I_y = \frac{hb^3}{3}$$

$$I_{z0} = \frac{bh^3}{12}$$

$$I_{y0} = \frac{hb^3}{12}$$

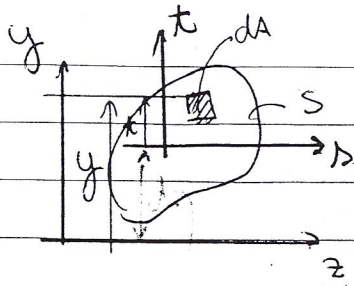
$$* I_z = \int_S y^2 dA = \int_0^h y^2 (b dy) \Rightarrow I_z = b \int_0^h y^2 dy = b \left[\frac{y^3}{3} \right]_0^h = \frac{bh^3}{3}$$

$$* I_y = \int_S z^2 dA = \int_0^b z^2 (h dz) \Rightarrow I_y = h \int_0^b z^2 dz = \frac{hb^3}{3}$$

$$* I_{z0} = \int_S y_0^2 dA = \int_{-h/2}^{h/2} y_0^2 (b dy_0) \Rightarrow I_{z0} = b \int_{-h/2}^{h/2} y_0^2 dy_0 = b \left[\frac{y_0^3}{3} \right]_{-h/2}^{h/2} = b \left[\frac{h^3}{24} - \left(-\frac{h^3}{24} \right) \right]$$

$$\Rightarrow I_{z0} = \frac{bh^3}{12}$$

$$* I_{y0} = \int_S z_0^2 dA = \int_{-b/2}^{b/2} z_0^2 (h dz_0) \Rightarrow I_{y0} = h \int_{-b/2}^{b/2} z_0^2 dz_0 = \frac{b^3 h}{12}$$



$$y = t + d \quad (*)$$

$$I_z = \int_S y^2 dA \stackrel{(*)}{=} \int_S (t+d)^2 dA \Rightarrow I_z = \int_S (t^2 + 2td + d^2) dA \Rightarrow$$

$$I_z = \int_S t^2 dA + 2d \int_S t dA + d^2 \int_S dA$$

$$I_z = I_S + 2d \cdot M_{S_{0z}} + d^2 \cdot A \quad \text{Teorema de Steiner}$$

* ~~Se o eixo passar pelo baricentro~~ : Se o eixo passar pelo baricentro ($w \equiv z_0$):

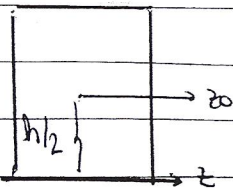
(baricentro)

$$I_z = I_{z_0} + 2d \cdot M_{S_{z_0}} + d^2 A$$

$$I_z = I_{z_0} + d^2 A$$

• Se tivermos várias figuras (no caso particular)

$$I_z^{\text{total}} = \sum_i I_z^i = \sum_i (I_{z_0}^i + d_i^2 A_i)$$

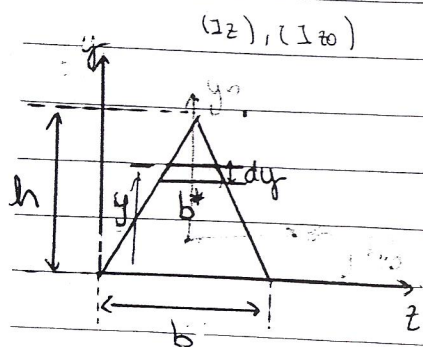


$$I_z = \frac{bh^3}{3}; A = b \cdot h, d = h/2$$

$$I_{z_0} = I_z - d^2 A = bh^3/3 - (h/2)^2 b \cdot h = bh^3 (1/3 - 1/4) = bh^3 \left(\frac{4-3}{12} \right) = \frac{bh^3}{12} \quad (v)$$



Exemplo:



(regra de 3 - semelhança)

$$\frac{b}{b^*} = \frac{h}{h-y} \rightarrow b^* = \left(\frac{h-y}{h}\right)b$$

aproximações do trapézio resa b) um retângulo \Rightarrow

$$\Rightarrow dA = b^* dy = \left(\frac{h-y}{h}\right)b dy$$

$$I_z = \int_S y^2 dA = \int_0^h y^2 \left(\frac{h-y}{h}\right)b dy$$

$$I_z = b \int_0^h y^2 dy - \frac{b}{h} \int_0^h y^3 dy = b \left[\frac{y^3}{3} \right]_0^h - \frac{b}{h} \left[\frac{y^4}{4} \right]_0^h$$

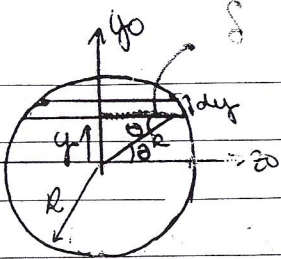
$$I_z = \frac{bh^3}{3} - \frac{bh^3}{4} \Rightarrow I_z = \frac{bh^3}{12}$$

Usando o teorema de Steiner.

$$I_{z0} = I_z - d^2A$$

$$I_{z0} = \frac{bh^3}{12} - \left(\frac{h}{3}\right)^2 \cdot \frac{bh}{3} = \frac{bh^3}{12} - \frac{bh^3}{12}$$

$$\Rightarrow I_{z0} = \frac{bh^3}{36} \Rightarrow I_{z0} = \frac{bh^3}{36}$$



$$\cos \theta = \frac{x}{R}, \quad \sin \theta = \frac{y}{R}$$

$$\Rightarrow \begin{cases} x = R \cos \theta \\ y = R \sin \theta \\ dy = R \cos \theta d\theta \end{cases}$$

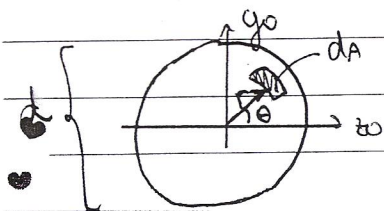
$$dA = 2x dy = 2R^2 \cos^2 \theta d\theta$$

* onde varia o θ \rightarrow

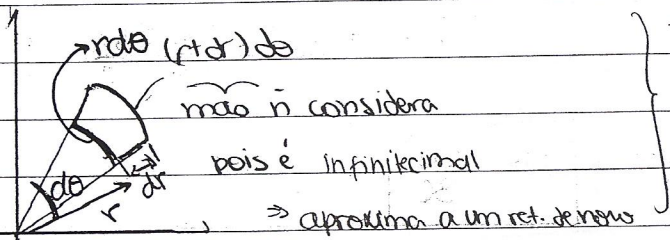
$$I_{z_0} = \int_S y^2 dA = \int_{-\pi/2}^{\pi/2} R^2 \sin^2 \theta (2R^2 \cos^2 \theta d\theta) = 2R^4 \int_{-\pi/2}^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta$$

$$\Rightarrow I_{z_0} = 2R^4 \left(\frac{\pi}{2} - \frac{3\pi}{8} \right) = \boxed{\frac{\pi R^4}{4}}$$

Momento polar de inércia



$$J_0 = \int_S r^2 dA$$



$$\Rightarrow dA = r^2 d\theta$$

$$\text{Logo, } J_0 = \int_S r^2 (r^2 d\theta) = \int_0^{2\pi} \int_0^R r^4 dr d\theta = 2\pi \int_0^R r^4 dr = 2\pi \left[\frac{r^5}{5} \right]_0^R$$

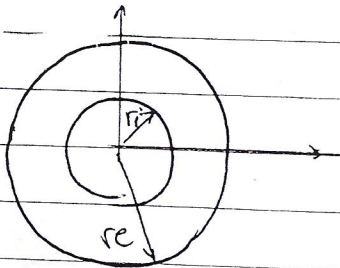
$$\Rightarrow J_0 = \frac{\pi R^5}{2}, \quad \text{já calculado antes em Torção}$$

$$r^2 = y^2 + z^2$$

$$J_0 = J_{z0} + J_{y0} \quad (J_0 \text{ é o total})$$

$$\text{por simetria, } J_{z0} = J_{y0} = J_0/2 = \pi R^4/4$$

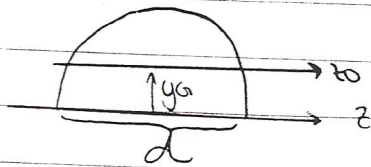
$$\text{em função do diâmetro } d: J_{z0} = J_{y0} = \frac{\pi d^4}{64}, \quad J_0 = \frac{\pi d^4}{32}$$



$$J_0 = \frac{\pi (r_e^4 - r_i^4)}{2}$$

$$J_0 = \frac{\pi (d_e^4 - d_i^4)}{32}$$

Exemplo: o momento de inércia de semi-círculo



$$y_G = \frac{M_{sz}}{A} \quad A = \frac{\pi d^2}{8}$$

$$dA = 2 \frac{d}{2} \cos \theta d\theta$$

$$= d \cos \theta \frac{d}{2} \cos \theta d\theta$$

(igual no ex anterior)

$$\Rightarrow dA = \frac{d^2}{2} \cos^2 \theta d\theta$$

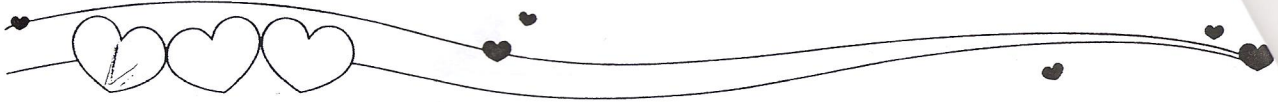
$$y = \frac{d}{2} \sin \theta$$

$$M_{sz} = \int y dA$$

$$\Rightarrow M_{sz} = \int_0^{\pi/2} \frac{d}{2} \sin \theta \left(\frac{d^2}{2} \cos^2 \theta d\theta \right)$$

$$M_{sz} = d^3/12$$

$$\text{Logo } y_G = \frac{d^3/12}{\pi d^2/8} = \frac{2d}{3\pi} //$$



$$I_z = \frac{\pi d^4}{64}, \quad J^{\text{circ}} = \frac{\pi d^4}{128}$$

Steiner: $I_{z0} = I_z + \text{dist}^2 \cdot A$

$$= \frac{\pi d^4}{128} + \left(\frac{2d}{3\pi}\right)^2 \frac{\pi d^2}{8}$$

$$I_{z0} \approx 0.00686d^4$$

28/05/2015

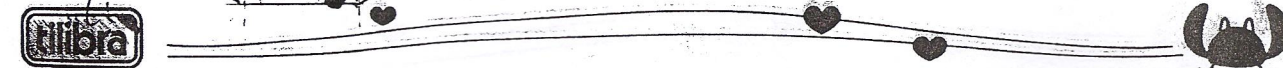
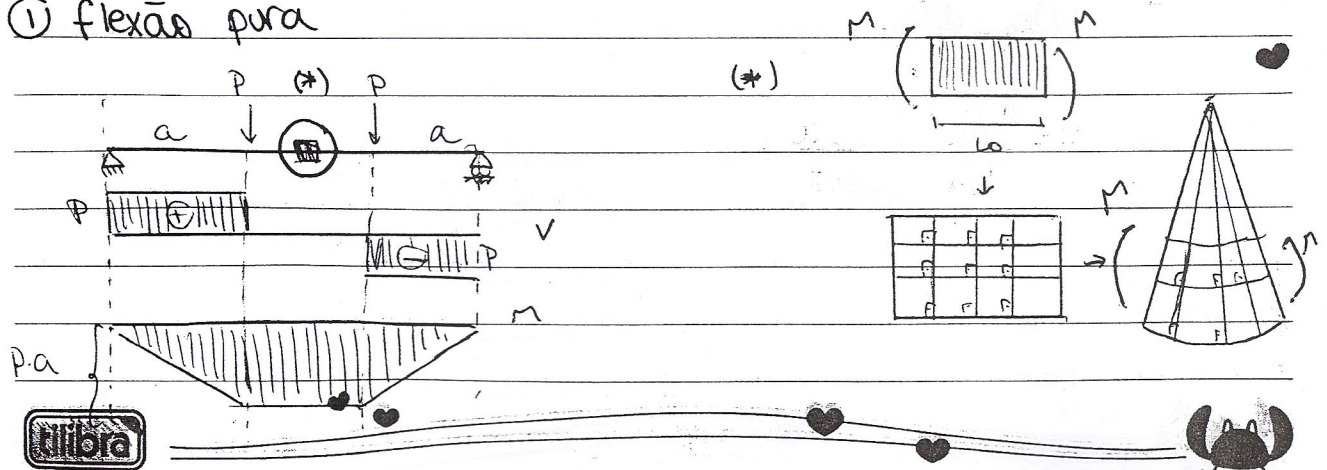
Tensões na Flexão

$N \Rightarrow \sigma = \frac{N}{A}$ $T \Rightarrow \tau = \frac{T}{J_0}$ $M = ?$

① * flexão pura $\left\{ \begin{array}{l} N=0 \\ V=0 \\ M \neq 0 \end{array} \right.$

② * flexão simples $\left\{ \begin{array}{l} N=0 \\ V \neq 0 \\ M \neq 0 \end{array} \right.$

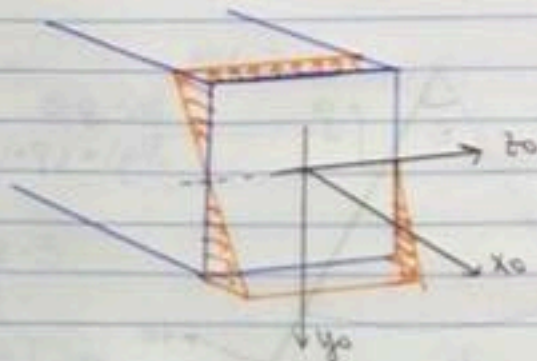
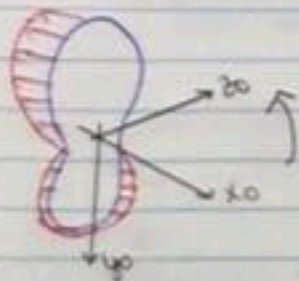
① flexão pura



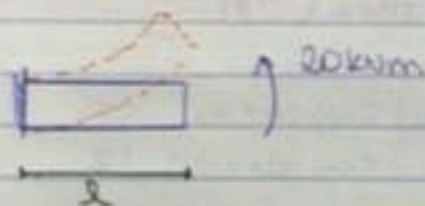
... substituindo em 1:

$$\sigma = \frac{M}{I} y$$

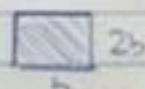
fórmula da tensão normal na flexão
 mom. de inércia



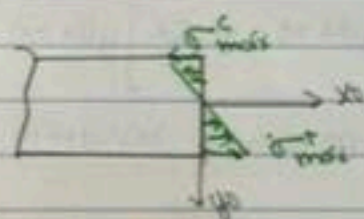
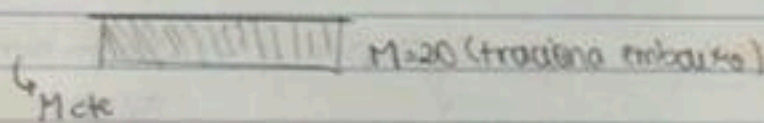
Exemplo



seção transversal (ST)



Determinar b mínimo p/ $\bar{\sigma} = 60 \text{ MPa}$



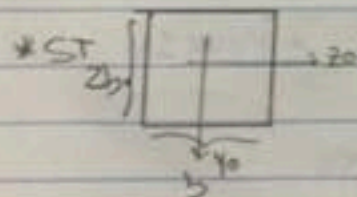
$$\sigma = \frac{M}{I} y$$

$$\sigma_{\text{max}} = \left| \frac{M}{I} y_{\text{max}} \right|$$

$$I = \frac{b(2b)^3}{12} = \frac{8b^4}{12} = \frac{2}{3} b^4$$

$$\sigma_{\text{max}} = \frac{M}{\frac{2}{3} b^4} \cdot b = \frac{3M}{2b^3}$$

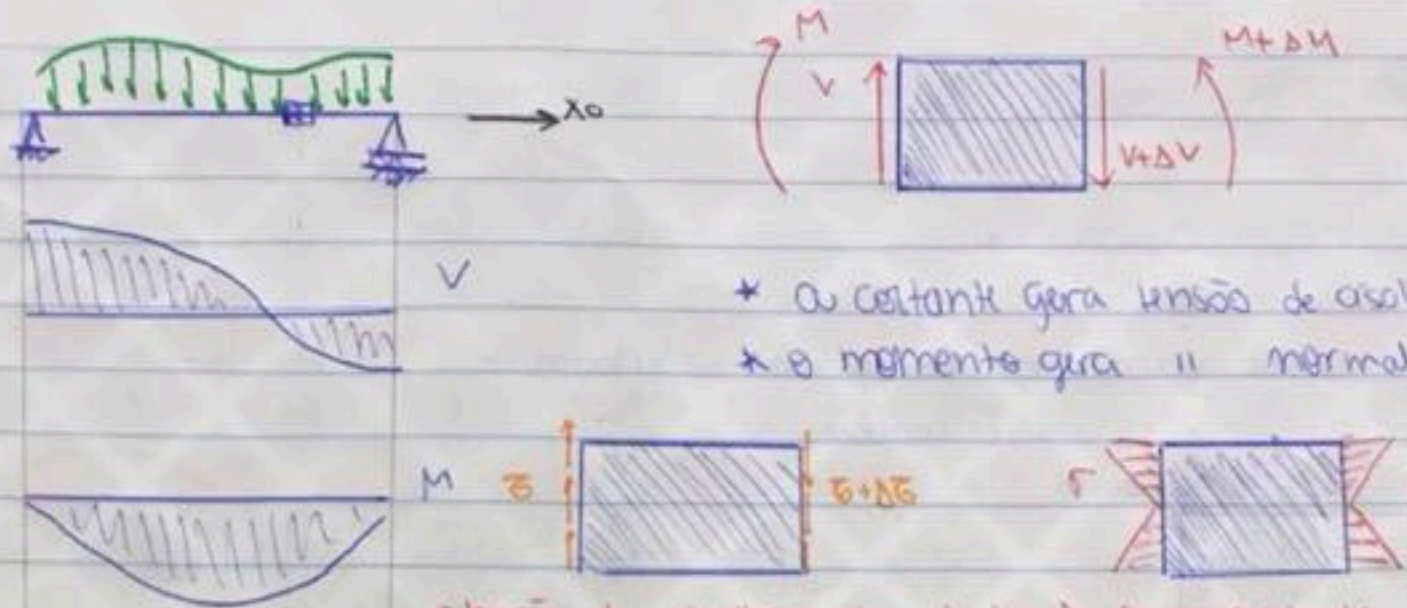
$$\frac{3M}{2b^3} \leq \bar{\sigma} = \frac{2b^3}{3M} \geq \frac{1}{\bar{\sigma}}$$



$$\Rightarrow b_{\text{min}} \geq \sqrt[3]{\frac{3M}{2\bar{\sigma}}} \Rightarrow b_{\text{min}} \geq \sqrt[3]{\frac{3 \cdot 20 \cdot 10^3}{2 \cdot 60 \cdot 10^6}} \Rightarrow b_{\text{min}} \geq 7,079 \text{ cm}$$

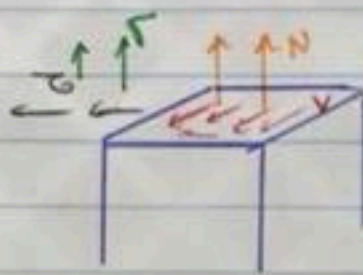
$$b_{\text{min}} = 7,9 \text{ cm} \approx 8 \text{ cm}$$

2) Flexão Simples

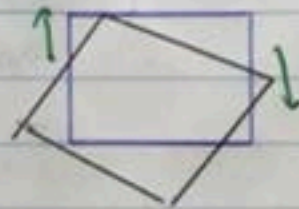


- * A constante gera tensão de cisalhamento
- * o momento gera a normal

tensão de cisalhamento devido à força cortante
 $\tau = \frac{V}{A} \rightarrow$ área da seção transversal



+ cisalhamento: \rightarrow distorção

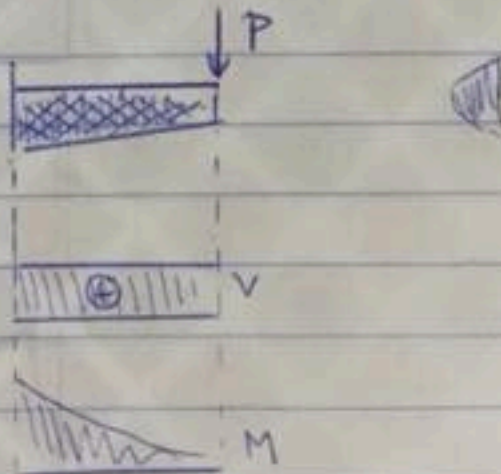


Hipótese de Navier

- + Vigas esbeltas (comprimento \gg dimensões da seção transversal)
- + Seção plana permanece plana

$$\sigma(x,y) = \frac{M(x)}{I_{2D}(x)} \cdot y$$

Exemplo



Tensões na Flexão

1) FLEXÃO PURA

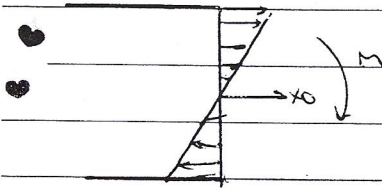
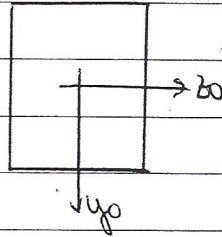
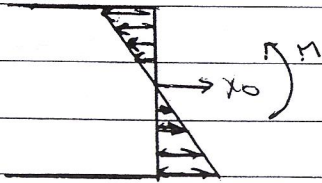
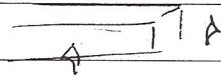
$$N=0, V=0; M = cte$$



- seções permanecem planas após a deformação
- a linha neutra é concêntrica

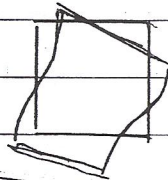
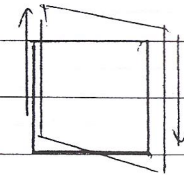
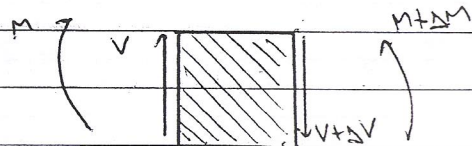
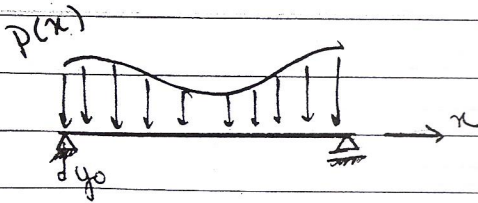
tensão = 0 ?

$$\sigma = \frac{M \cdot y}{I_{z0}}$$



2) FLEXÃO SIMPLIS

$$N=0, V \neq 0, M \neq cte$$

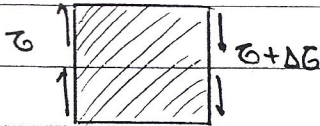


continuação folha

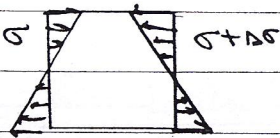


implicam:

- tensões normal devido a M
- tensões de cisalhamento devido a V



$$\bar{\sigma} = \int v da$$



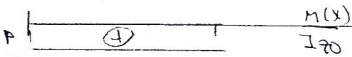
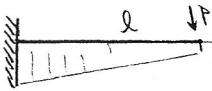
$$\sigma(x,y)$$

seção transv.

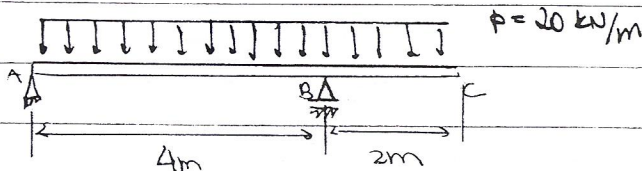
Hipótese de Navier (só vale p/ vigas esbeltas: comprimento \gg dimensões da ST)

Seções que eram planas e perpendiculares aos eixos undeformados da barra permanecem planas e perpendiculares aos eixos deformados

$$\sigma(x,y) = \frac{M(x)}{I_{z0}(x)} \cdot y$$

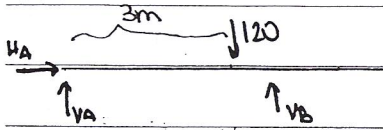


Exemplo: calcular o coef de segurança na estrutura a seguir (dados: limite de ruptura σ_{ST})



Reações de apoio

DCL



equilíbrio

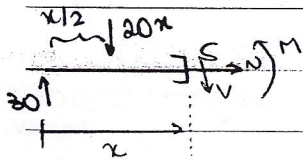
$$\sum F_H = 0 \Rightarrow H_A = 0$$

$$\sum F_V = 0 \Rightarrow V_A + V_B = 120$$

$$\sum M_A = 0 \Rightarrow -20 \cdot 3 + V_B \cdot 4 = 0$$

$$\Rightarrow \begin{cases} V_B = 90 \text{ kN} \\ V_A = 30 \text{ kN} \end{cases}$$

Trecho AB

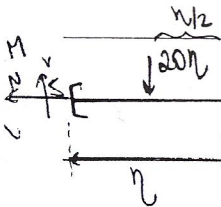


$$\sum F_H = 0 \Rightarrow N = 0$$

$$\sum F_V = 0 \Rightarrow 30 - 20x - V = 0 \Rightarrow V(x) = 30 - 20x$$

$$\sum M_S = 0 \Rightarrow -M - 30x + 20x \cdot \frac{x}{2} = 0 \Rightarrow M(x) = 30x - 10x^2$$

Trecho BC

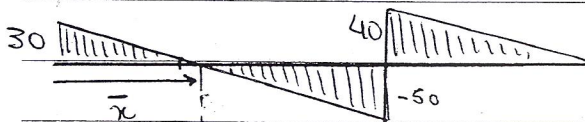


$$\sum F_H = 0 \Rightarrow N = 0$$

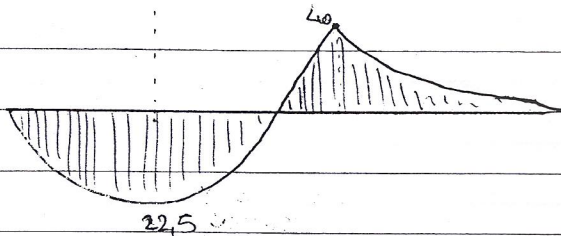
$$\sum F_V = 0 \Rightarrow V - 20\eta = 0 \Rightarrow V(\eta) = 20\eta$$

$$\sum M_S = 0 \Rightarrow -M - 20\eta \cdot \frac{\eta}{2} = 0 \Rightarrow M(\eta) = -10\eta^2$$

diagramas



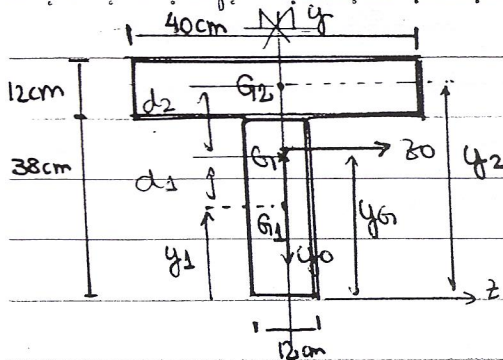
$$\bar{x} = 1,5 \text{ m}$$



$$M(\bar{x}) = 22,5 \text{ kNm}$$

$\sigma = 17 \text{ OK}$
 170
 (achar)

* Considere a seguinte ST:



$$I_{z0} = ?$$

$$y_G = \frac{\sum y_i A_i}{\sum A_i}$$

$$y_1 = 19 \text{ cm} \quad y_2 = 44 \text{ cm}$$

$$A_1 = 12 \cdot 38 = 456 \text{ cm}^2$$

$$A_2 = 12 \cdot 40 = 480 \text{ cm}^2$$

$$y_G = \frac{y_1 A_1 + y_2 A_2}{A_1 + A_2} = \frac{19 \cdot 456 + 44 \cdot 480}{456 + 480}$$

$$y_G = 31,8 \text{ cm}$$

$$I_{z0} = \sum I_{z0}^i = \sum (I_z^i + d_i^2 A_i)$$

$$d_2 = y_2 - y_G = 44 - 31,8 = 12,2 \text{ cm}$$

$$d_1 = y_G - y_1 = 31,8 - 19 = 12,8 \text{ cm}$$

$$I_z^1 = \frac{b_1 h_1^3}{12} = \frac{12 \cdot 38^3}{12} = 54872 \text{ cm}^4$$

$$I_z^2 = \frac{b_2 h_2^3}{12} = \frac{40 \cdot 12^3}{12} = 5760 \text{ cm}^4$$

$$\Rightarrow I_{z0} = [54872 + 12,8^2 \cdot 456] + [5760 + 12,2^2 \cdot 480] = 206786 \text{ cm}^4$$

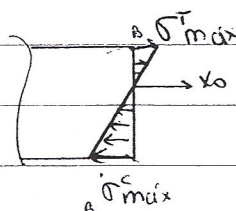
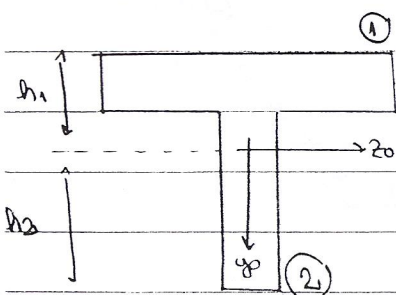
$$I_{z0} = 2,06786 \cdot 10^5 \cdot 10^{-8} \text{ m}^4 \Rightarrow$$

$$I_{z0} = 2,07 \cdot 10^{-3} \text{ m}^4$$

* Considere também as seguintes tensões limite $\sigma_{R^C} = 30 \text{ MPa}$ e $\sigma_{R^T} = 5 \text{ MPa}$ Qual o coef. de segurança?

(pegar o momento mínimo)

seção B $M_B = -40 \text{ kNm}$



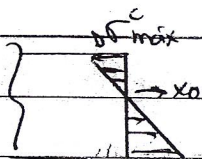
$$h_1 = 50 - y_G = 50 - 18,2 \text{ cm} = 31,8 \text{ cm}$$

$$h_2 = y_G = 31,8 \text{ cm} = 31,8 \text{ cm}$$

$$\sigma^T = \frac{M_B (-h_1)}{I_{z0}} = \frac{-40 \cdot 10^3 (-0,182)}{2,07 \cdot 10^3} = 3,5 \text{ MPa}$$

$$\sigma^C = \frac{M_B (h_2)}{I_{z0}} = \frac{-40 \cdot 10^3 (0,318)}{2,07 \cdot 10^3} = -6,1 \text{ MPa}$$

seção D $M_D = 22,5 \text{ kNm}$ pego o mom. máximo



$$\sigma^T = \frac{M_D (-h_1)}{I_{z0}} = \frac{22,5 \cdot 10^3 (-0,182)}{2,07 \cdot 10^3} = -1,98 \text{ MPa}$$

$$\sigma^C = \frac{M_D (h_2)}{I_{z0}} = \frac{22,5 \cdot 10^3 (0,318)}{2,07 \cdot 10^3} = 3,46 \text{ MPa}$$

Portanto,

$$\sigma^T_{\text{máx}} = \text{máx} (\sigma^T_{\text{máx}}, \sigma^T_{\text{máx}}) = 3,5 \text{ MPa}$$

$$\sigma^C_{\text{máx}} = -6,1 \text{ MPa}$$

$$\bar{\sigma} = \frac{\sigma_{\text{lim}}}{S} \quad \sigma^T_{\text{máx}} = \frac{\sigma_{\text{R}}^T}{S_T} \Rightarrow S_T = \frac{\sigma_{\text{R}}^T}{\sigma^T_{\text{máx}}} = \frac{5}{3,5} = 1,4$$

$$S_C = \frac{\sigma_{\text{R}}^C}{|\sigma^C_{\text{máx}}|} = \frac{30}{6,1} = 4,9$$

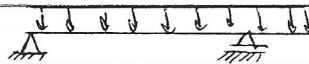
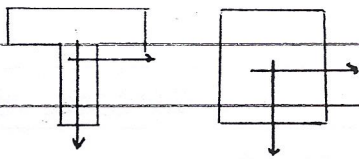
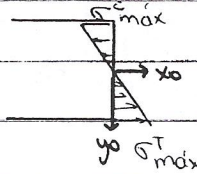
$$S = \min(S_T, S_C) = 1,4$$

Tensão na Flexão

- flexão simples

* σ é nula no baricentro (a linha neutra é baricêntrica)

$$\sigma = \frac{M \cdot y}{I_{z0}}$$



Sempre dobrado?

σ_R^T, σ_R^C

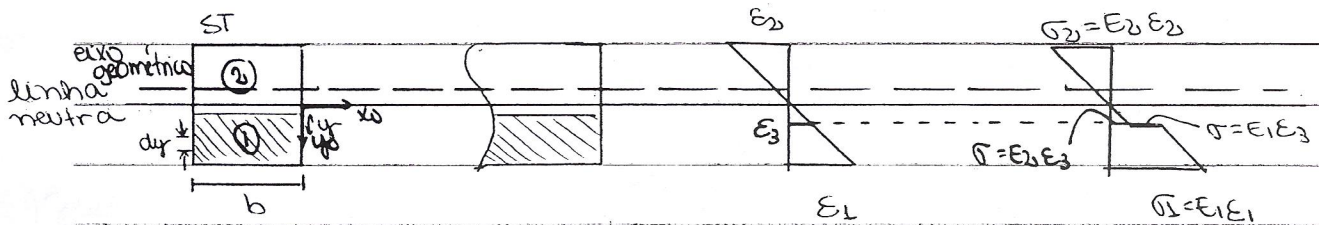
* reações de apoio

* diagramas

* propriedades da seção

* cálculo das tensões

Diferentes Materiais



$$\epsilon = \kappa y$$

$$\sigma = E \epsilon \text{ (Lei de Hooke)}$$

$$\sigma(y) = E(y) \epsilon(y) = \kappa E(y) y$$

$$N = \int \sigma dA = 0 \Rightarrow \int \kappa E(y) y \cdot b \cdot dy = 0$$

$$\Rightarrow \kappa E_{eq} \int \left(\frac{E(y) b}{E_{eq}} \right) y dy = 0$$

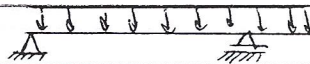
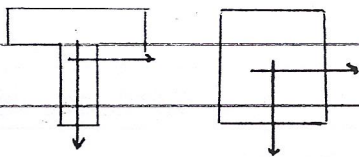
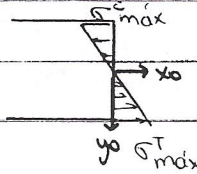
$$\kappa E_{eq} \int b_{eq} y dy = 0 \Rightarrow \kappa E_{eq} \int y dA_{eq} = 0$$

Tensão na Flexão

- flexão simples

* σ é nula no baricentro (a linha neutra é baricêntrica)

$$\sigma = \frac{M \cdot y}{I_{z0}}$$



Sempregado?

σ_R^T, σ_R^C

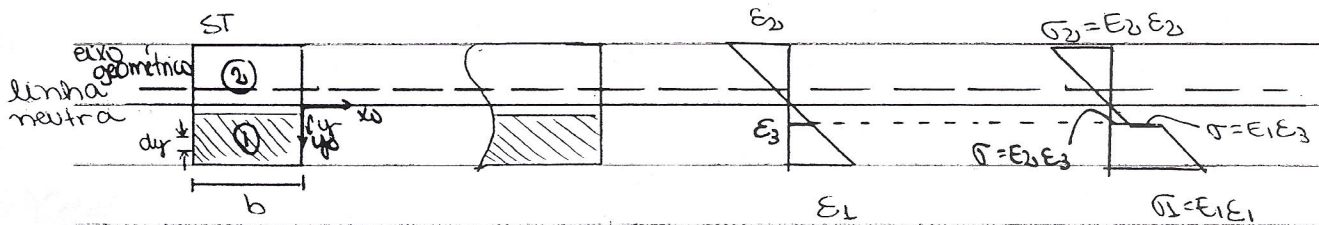
* reações de apoio

* diagramas

* propriedades da seção

* cálculo das tensões

Diferentes Materiais



$$\epsilon = k \cdot y$$

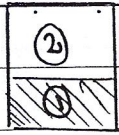
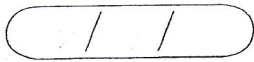
$$\sigma = E \cdot \epsilon \quad (\text{Lei de Hooke})$$

$$\sigma(y) = E(y) \cdot \epsilon(y) = k \cdot E(y) \cdot y$$

$$N = \int \sigma \cdot dA = 0 \Rightarrow \int k \cdot E(y) \cdot y \cdot b \cdot dy = 0$$

$$\Rightarrow k \cdot E_{eq} \int \left(\frac{E(y)}{E_{eq}} \right) y \cdot dy = 0$$

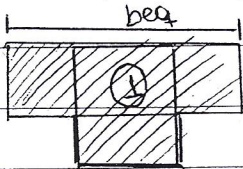
$$k \cdot E_{eq} \int b \cdot eq \cdot y \cdot dy = 0 \Rightarrow k \cdot E_{eq} \int y \cdot dA_{eq} = 0$$



$E_{eq} = E_1$ ① (igualar os módulos de elasticidade
 e compensar no comprimento \Rightarrow usar flex. simpl.)
 $E_{eq} = E_2$ ②

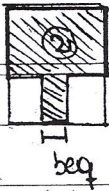
$E_2 > E_1$

① $E_{eq} = E_1$



$b_{eq} = \frac{E_2 \cdot b}{E_1} > b$

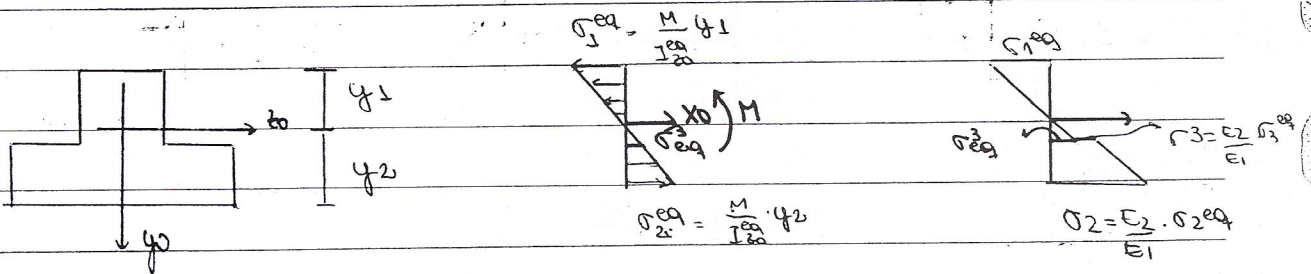
② $E_{eq} = E_2$



$b_{eq} = \frac{E_1 \cdot b}{E_2} < b$

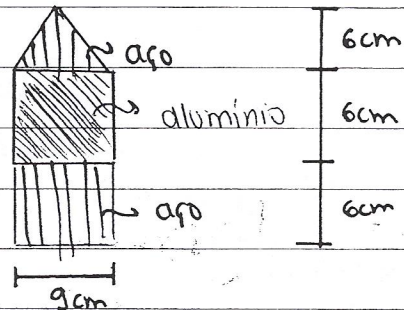
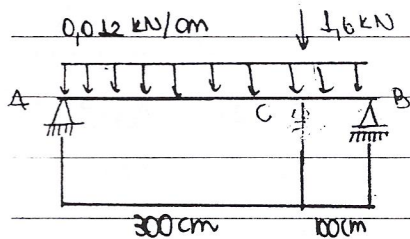
$y_G = \frac{\sum A_i y_{Gi}}{\sum A_i}$

$I_{z0}^{eq} = \sum I_{z0}^i + d_i^2 A_i$



① E_1

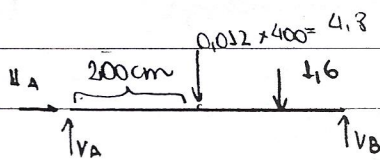
Exercício: Determinar as tensões máximas no aço e no alumínio



$E_{aço} = 3E_{al}$

Reações de Apoio

DCL



$\sum F_H = 0 \quad H_B = 0$

$\sum F_V = 0 \quad V_A + V_B = 1,6 + 4,8 = 6,4$

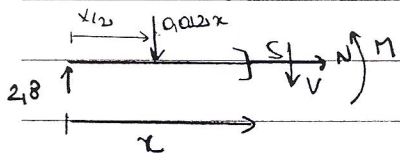
$\sum M_A = 0 \quad -4,8 \times 200 - 1,6 \times 400 + V_B \times 400 = 0$

$V_B = 3,6 \text{ kN}$

$\Rightarrow V_A = 2,8 \text{ kN}$

Diagramas

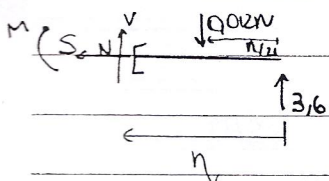
Trecho (AC) $0 < x < 300 \text{ cm}$ $N=0$



$\sum F_V = 0 \quad 2,8 - 0,012x - V = 0 \Rightarrow V = 2,8 - 0,012x$

$\sum M_S = 0 \quad M - 2,8x + 0,012x \cdot \frac{x}{2} = 0 \Rightarrow M = 2,8x - 0,006x^2$

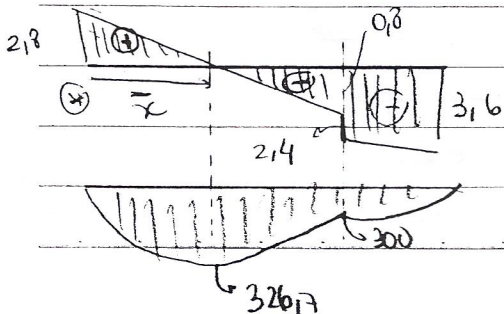
Trecho BC $0 < \eta < 100 \text{ cm}$



$N=0$

$\sum F_V = 0 \quad V - 0,012\eta + 3,6 = 0 \Rightarrow V = 0,012\eta - 3,6$

$\sum M_S = 0 \quad M - 0,012\eta \cdot \frac{\eta}{2} + 3,6\eta = 0 \Rightarrow M = 3,6\eta - 0,006\eta^2$

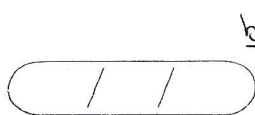


$V \text{ [kN]}$

$\bar{x} = 233,33 \text{ cm}$

$M(\bar{x}) = 226,17 \text{ kNcm}$

$M \text{ [kNcm]}$



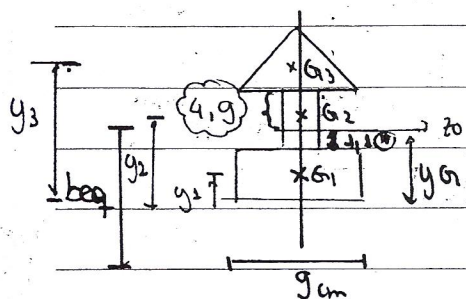
$$\frac{bh^3}{12} \quad \square \quad h$$

$$\frac{bh^3}{36} \quad \triangle \quad h$$

$$\frac{\pi d^4}{4}$$

Escolhendo $E_{eq} = E_{aço}$

$$b_{eq} = \frac{E_{al}}{E_{aço}} \cdot b = \frac{b}{3} = 30 \text{ cm}$$



$$y_1 = 3 \text{ cm}$$

$$A_1 = 9 \times 6 = 54 \text{ cm}^2$$

$$y_2 = 9 \text{ cm}$$

$$A_2 = 3 \times 6 = 18 \text{ cm}^2$$

$$y_3 = 14 \text{ cm}$$

$$A_3 = \frac{9 \times 6}{2} = 27 \text{ cm}^2$$

$$y_{c1} = \frac{\sum A_i y_i}{\sum A_i} = \frac{54 \times 3 + 18 \times 9 + 27 \times 14}{54 + 18 + 27}$$

$$I_{z0}^{eq} = \sum I_z^i + d_i^2 A_i$$

$$d_1 = |y_{c1} - y_1| = y_{c1} - y_1$$

$$d_2 = |y_{c1} - y_2| = y_2 - y_{c1}$$

$$d_3 = |y_{c1} - y_3| = y_3 - y_{c1}$$

$$\Rightarrow I_z^1 = \frac{9 \times 6^3}{12} \quad I_z^2 = \frac{3 \times 6^3}{12} \quad I_z^3 = \frac{9 \times 6^3}{36}$$

$$\Rightarrow I_{z0}^{eq} = [I_z^1 + d_1^2 A_1] + [I_z^2 + d_2^2 A_2] + [I_z^3 + d_3^2 A_3] = 2528,2 \text{ cm}^4$$

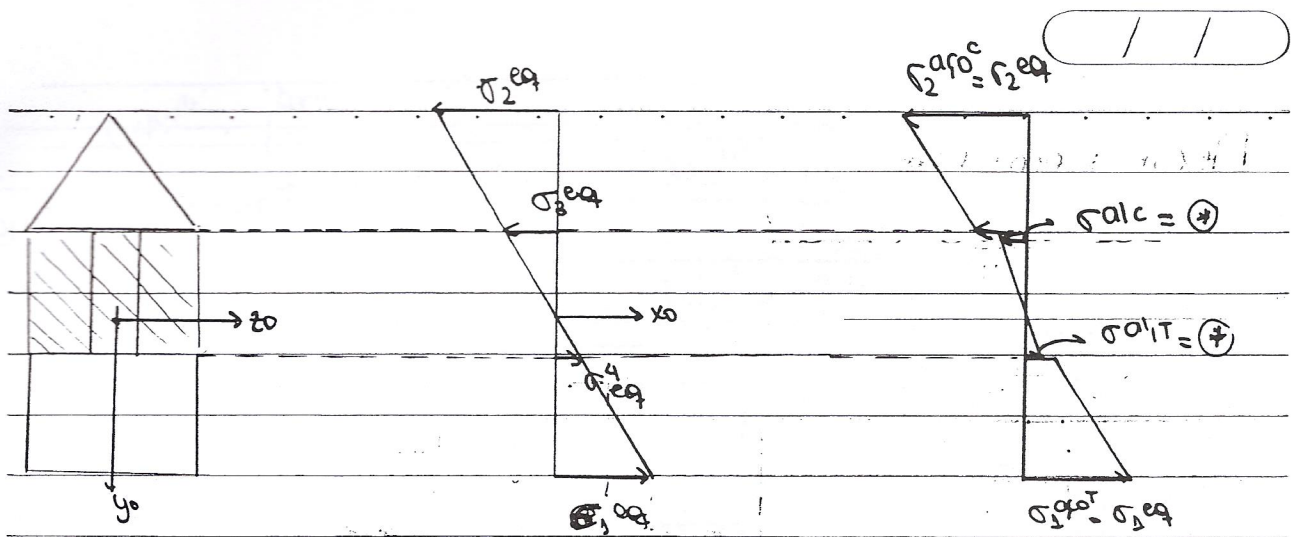
Pl ou seção de maior momento ($M = 326,7 \text{ kNcm}$)

$$\sigma_1^{eq} = \frac{326,7}{2528,2} = 0,129 \text{ kN/cm}^2 = 0,129 \frac{10^3 \text{ N}}{10^4 \text{ cm}^2} \Rightarrow \sigma_1^{eq} = 9,2 \text{ MPa}$$

$$\sigma_2^{eq} = \frac{326,7}{2528,2} (-10,9) = -1,41 \text{ kN/cm}^2 = -14,1 \text{ MPa}$$

$$\sigma_3^{eq} = \frac{326,7}{2528,2} (-4,9) = -6,3 \text{ MPa}$$

$$\sigma_4^{eq} = \frac{326,7}{2528,2} (2,1) = 1,4 \text{ MPa}$$



$$(*) \quad \sigma_{alc} = \frac{E_{al}}{E_{aso}} \cdot \sigma_3^{eq} = \frac{1}{3} (-6,3) = -2,1 \text{ MPa}$$

$$\sigma_{alt} = \frac{1}{3} \cdot \sigma_4^{eq} = \frac{1,4}{3} = 0,47 \text{ MPa}$$

Obs $\sigma^{aso} = E_{aso} \cdot \epsilon$ $\frac{\sigma_{aso}}{E_{aso}} = \frac{\sigma_{al}}{E_{al}} \Rightarrow \sigma_{al} = E_{al} \frac{\sigma_{aso}}{E_{aso}}$

$\sigma_{al} = E_{al} \cdot \epsilon$

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Flexão normal simples

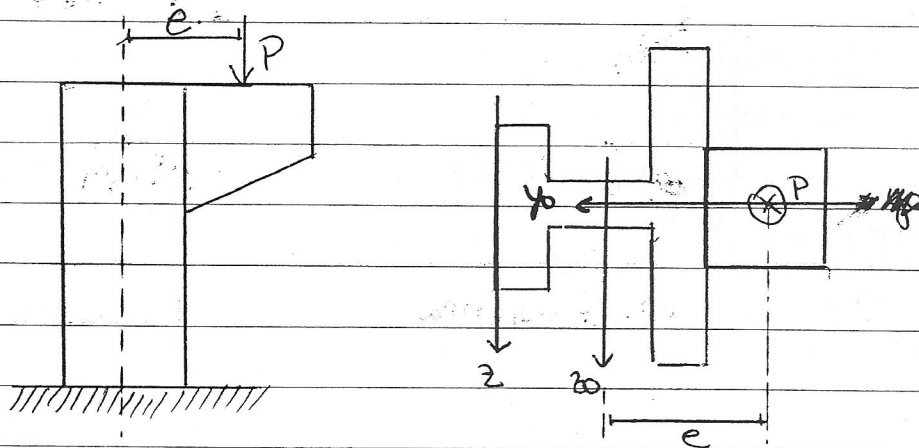
• flexão pura $\sigma = \frac{M}{I_{z0}} \cdot y$

• hipótese de Navier $\sigma(x,y) = \frac{M(x)}{I_{z0}(x)} \cdot y$

• diferentes materiais $E_{eq} \Rightarrow b_{eq}$
 \hookrightarrow atualizar as dimensões $\sigma = \frac{E_2}{E_1} \cdot \sigma_{eq}$

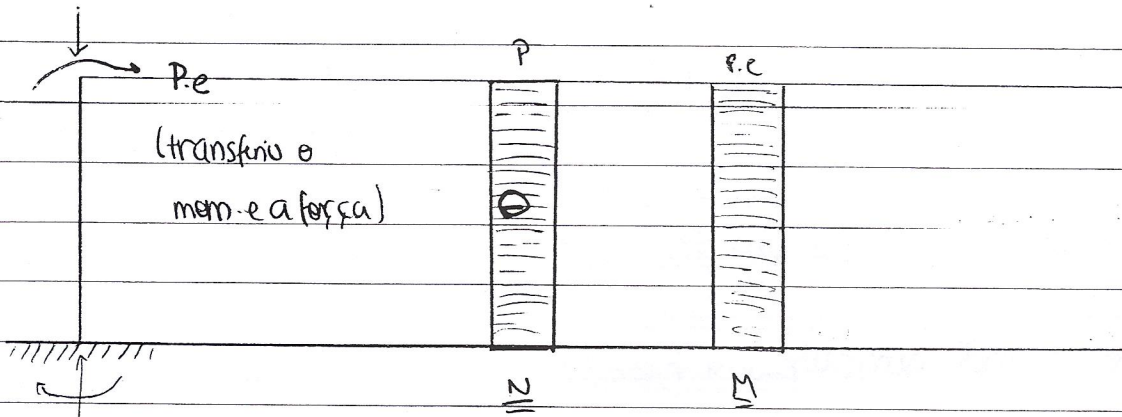
Flexão Normal Composta

$$N \neq 0 \quad V \neq 0 \quad M \neq 0$$

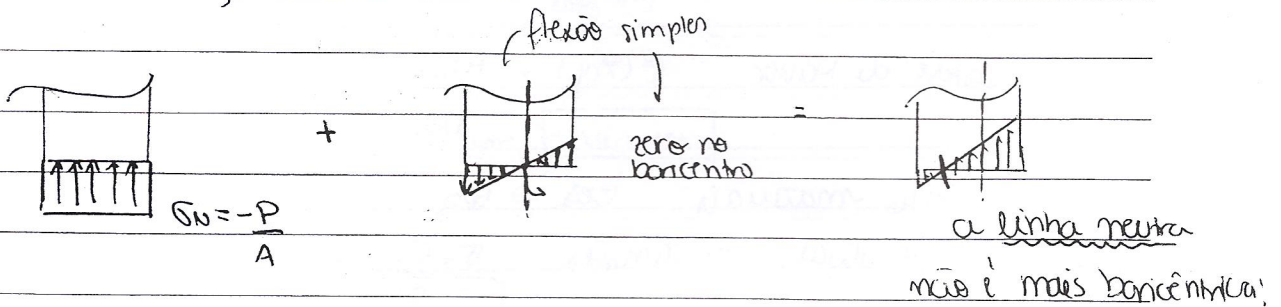


pilar visto de lado

pilar visto de cima



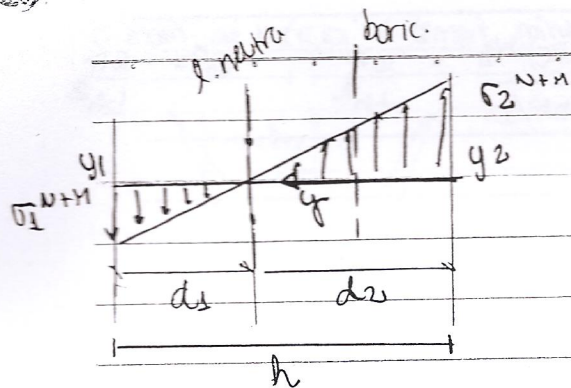
duas contribuições na tensão



tensão na flexão normal composta

A tensão é: $\sigma = \sigma_0 + \sigma_M$

$$\sigma = \frac{N}{A} + \frac{M}{I_{y0}} \cdot y$$



$$\sigma_1^{N+M} = -\frac{P}{A} + \frac{(Pe)y_1}{I_{z0}}$$

$$\sigma_2^{N+M} = -\frac{P}{A} - \frac{(Pe)y_2}{I_{z0}}$$

Encontramos a nova posição da linha neutra por semelhança de triângulos

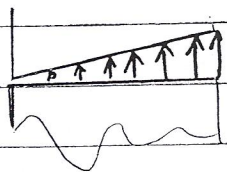
$$\frac{\sigma_2 - \sigma_1}{h} = \frac{\sigma_1}{d_1} = \frac{\sigma_2}{d_2}$$

Condições de projeto

$$\begin{cases} \bar{\sigma}_c \leq \sigma_1 \leq \bar{\sigma}_t \\ \bar{\sigma}_c \leq \sigma_2 \leq \bar{\sigma}_t \end{cases}$$

* Importam os sinais!

Caso interessante para o exemplo $\bar{\sigma}_t = 0$

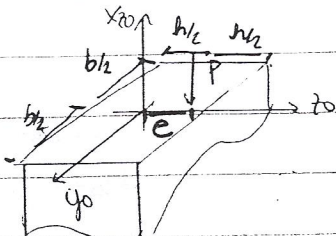


$$\sigma_1 \leq \bar{\sigma}_t = 0$$

$$-\frac{P}{A} + \frac{(Pe)y_1}{I_{z0}} \leq 0$$

$$e \leq \frac{I_{z0}}{Ay_1}$$

seção retangular



$$\sigma^N = \frac{-P}{A} = \frac{-P}{b \cdot h}$$

$$I_{z0} = \frac{bh^3}{12}$$

$$\sigma_1^M = \frac{(Pe) \cdot h/2}{bh^3/12} = \frac{6Pe}{bh^2}$$

$$\sigma_2^M = -\frac{6Pe}{bh^2}$$

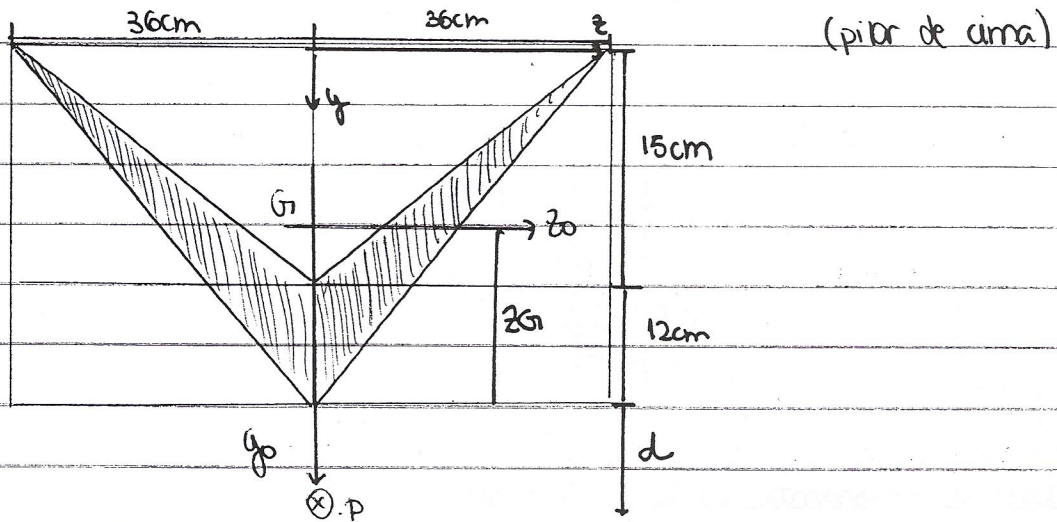
Para não haver tração: $\sigma_1 = \sigma^N + \sigma_1^M \leq 0$

$$\frac{-P}{bh} + \frac{6Pe}{bh^2} \leq 0$$

$$-1 + \frac{6e}{h} \leq 0$$

$$e \leq \frac{h}{6}$$

Exemplo: Para um pilar com ST dada a seguir, encontra d para que $|\sigma_{\max}^T| = |\sigma_{\max}^C|$



$$A_{TOT} = A_T = A_1 - A_2 = \frac{36 \cdot 15}{2} - \frac{26 \cdot 15}{2} = 972 - 540$$

$$A_T = 432 \text{ cm}^2$$

$$y_{G1} = \frac{27}{3} = 9 \text{ cm}$$

$$y_{G2} = \frac{15}{3} = 5 \text{ cm}$$

$$y_G = \frac{A_1 y_{G1} - A_2 y_{G2}}{A_T} = \frac{972 \cdot 9 - 540 \cdot 5}{432}$$

$$y_G = 14 \text{ cm}$$

$$I_{z0} = I_z^{(1)} - I_z^{(2)}$$

$$I_z^{(1)} = I_{z0}^{(1)} + d_1^2 \cdot A_1$$

$$I_z^{(2)} = I_{z0}^{(2)} + d_2^2 \cdot A_2$$

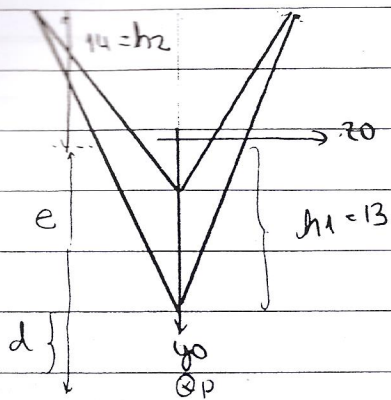
$$I_{z0}^{(1)} = \frac{bh^3}{36}$$

$$d_1 = |y_1 - y_G| = 5 \text{ cm}$$

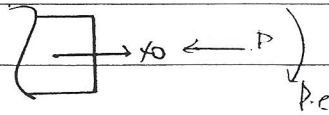
$$d_2 = |y_2 - y_G| = 9 \text{ cm}$$

$$\Rightarrow I_{z0} = 8176 \text{ cm}^4$$

O mom de inercia é sempre mínimo no baricentro



$$e = 13 + d$$



$$\sigma_1^{N+M} = \frac{-P}{A} - \frac{(Pe) \cdot h_1}{I_{z0}} \quad e \quad \sigma_2^{N+M} = \frac{-P + (Pe) \cdot h_2}{A \cdot I_{z0}}$$

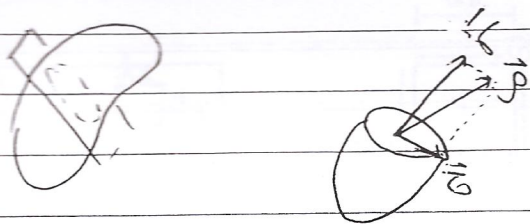
> 0

$$\sigma_{max} = \sigma_2 = |\sigma_1| - |\sigma_{max}|$$

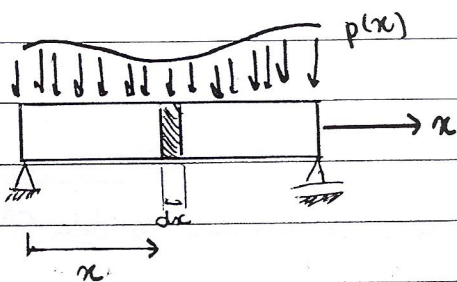
$$\frac{-P + (Pe) \cdot h_2}{A \cdot I_{z0}} = \frac{P + (Pe) \cdot h_1}{A \cdot I_{z0}} \Rightarrow \frac{e(h_2 - h_1)}{I_{z0}} = \frac{2P}{A} \Rightarrow e = \frac{2I_{z0}}{A(h_2 - h_1)}$$

$$e = 61 \text{ cm} \quad d = e - 13 \Rightarrow d = 61 - 13 = \underline{48 \text{ cm}}$$

Cisalhamento na Flexão

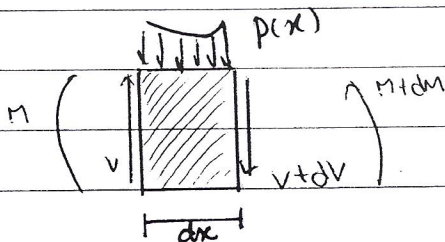


Lembrando:



* a relação entre o carregamento aplicado e a força cortante: $\frac{dV}{dx} = -p$

* a relação entre força cortante e momento fletor: $\frac{dM}{dx} = V$



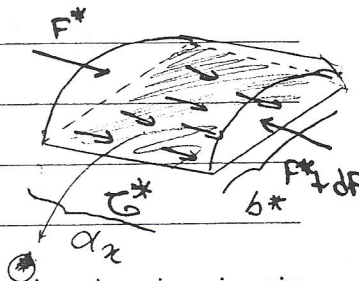
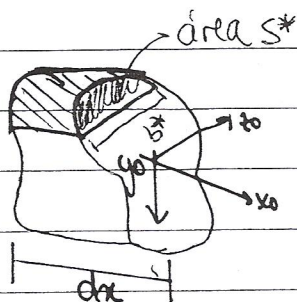
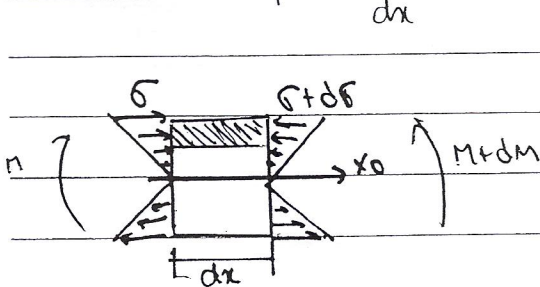
Integrando as equações para obter as expressões solicitadas:

$$N = \int \sigma da$$

$$M = \int \sigma y da$$

$$V = \int \tau da$$

Assumindo que $\frac{dM}{dx} = V > 0$



//

Fazendo o Equilíbrio:

$$F^* - (F^* + dF) + \overline{G}^* (dx \cdot b^*) = 0$$

\uparrow área *

$$-dF + \overline{G}^* dx b^* = 0$$

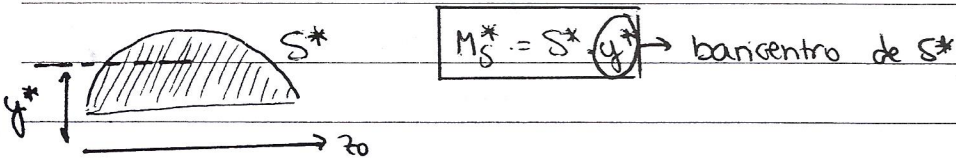
$$\boxed{\overline{G}^* = \frac{1}{b^*} \frac{dF}{dx}}$$

Mas

$$F^* = \int_{S^*} \sigma dA = \int_{S^*} \left(\frac{M}{I_{z0}} y \right) dA$$

$$F^* = \frac{M}{I_{z0}} \int_{S^*} y dA = \frac{M}{I_{z0}} M_S^*$$

momento estático da área S^*
em relação ao eixo z_0



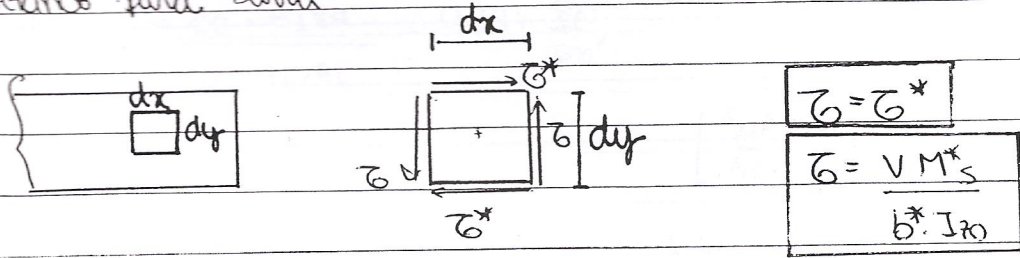
Calculando dF/dx :

$$\frac{dF}{dx} = \frac{d}{dx} \left(\frac{M}{I_{z0}} M_S^* \right) = \frac{M_S^*}{I_{z0}} \frac{dM}{dx} = \frac{M_S^*}{I_{z0}} \cdot V$$

Substituindo

$$\boxed{\overline{G}^* = \frac{V \cdot M_S^*}{b^* \cdot I_{z0}}}$$

Volando para baixo

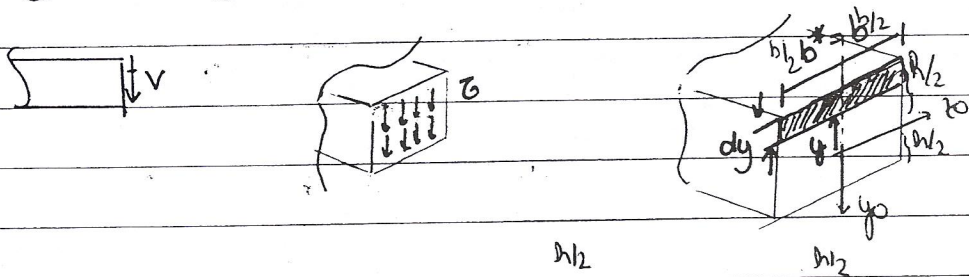


Fórmulas : (já vistas) $\tau = N/A$; $\tau = \frac{M}{I_{z0}} y$; $\tau = \frac{T}{J_0} r$ (circular)
 $\tau = \frac{VM_s^*}{b^* I_{z0}}$

$$V = \int \tau da \Rightarrow V = \int \frac{VM_s^*}{I_{z0} b^*} da = \frac{V}{I_{z0}} \int \frac{M_s^*}{b^*} da$$

Podemos provar que $I_{z0} = \int \frac{M_s^*}{b^*} da$

Seção Retangular



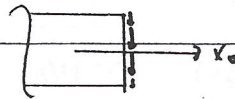
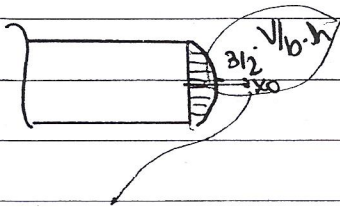
$$I_{z0} = \frac{bh^3}{12}$$

$$M_s^* = \int y da = \int_y^{h/2} y \cdot b^* dy = b \int_y^{h/2} y dy$$

$$M_s^* = b \left[\frac{y^2}{2} \right]_y^{h/2} = \frac{b}{2} \left[\frac{h^2}{4} - y^2 \right] = \frac{bh^2}{8} \left[1 - \frac{4y^2}{h^2} \right]$$

$$\tau = \frac{VM_s^*}{b^* I_{z0}} = \frac{V \cdot bh^2}{8} \left[\frac{1-4y^2}{h^2} \right] \cdot \frac{1}{b \frac{bh^3}{12}}$$

$$\tau = \frac{3}{2} \frac{V}{bh} \left[\frac{1-4y^2}{h^2} \right]$$



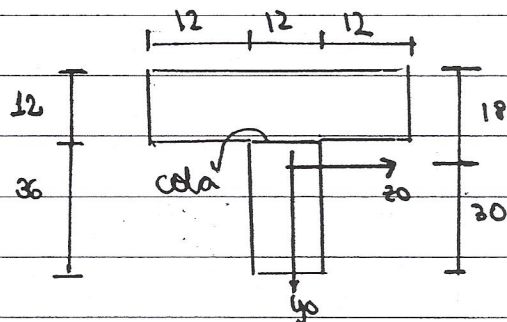
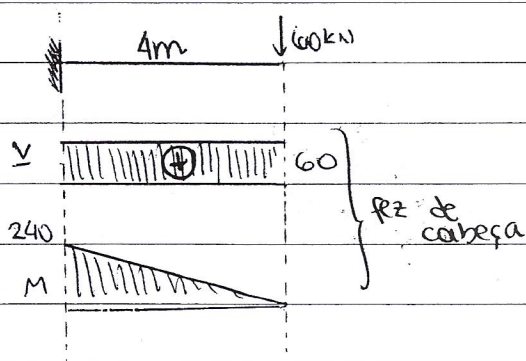
$$\tau_{\text{média}} = \frac{V}{b \cdot h}$$

$$\tau_{\text{máx}} = f \tau_{\text{média}}$$

f = fator de concentração

f = 3/2 para este caso

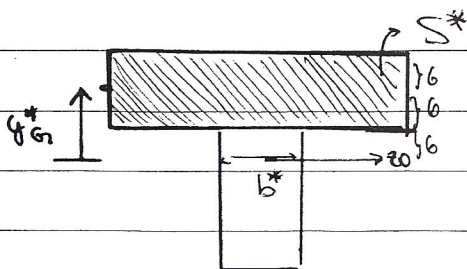
Exemplo: Determinar a máxima tensão de ruptura de um trabalho na cola para que o fator de segurança seja 2.



$$I_{z0} = 176256 \text{ cm}^4$$

$$f_{\sigma} = \frac{\sigma - \sigma_R}{\sigma}$$

$$f_{\tau} = \frac{\tau - \tau_R}{\tau}$$



$$y^* = 12 \text{ cm}$$

$$b^* = 12 \text{ cm}$$

$$S^* = 12 \times 36 \text{ cm}^2$$

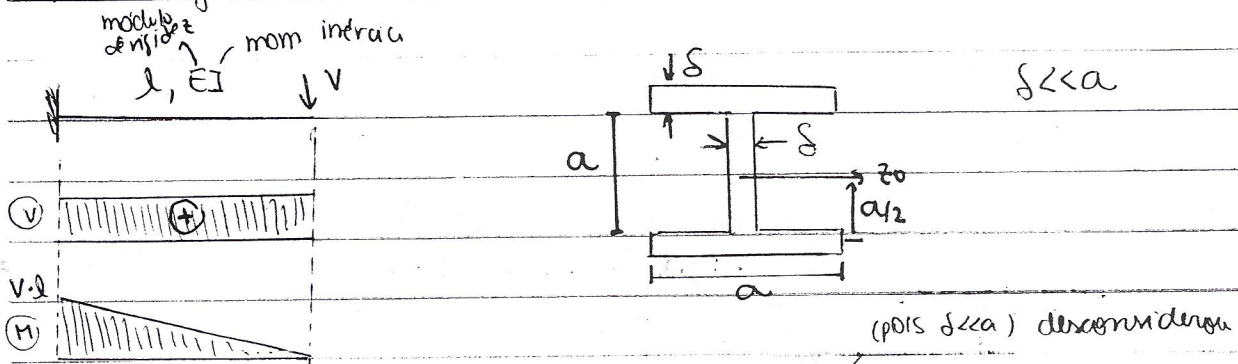
$$M_s^* = S^* \cdot y^* = (12 \times 36) \times 12 = 5184 \text{ cm}^3$$

$$\bar{\sigma} = \frac{VM^*}{b^* J_{z0}} = \frac{60 \cdot 5184}{12 \cdot 176256} = 0,147 \frac{\text{kN}}{\text{cm}^2}$$

$$\bar{\sigma} = 1,47 \cdot \text{MPa} \approx 1,5 \text{ MPa}$$

limitante $\bar{\sigma} = \bar{\sigma}_R \Rightarrow \bar{\sigma}_R = \bar{\sigma} \cdot S \Rightarrow \bar{\sigma}_R \approx 3 \text{ MPa}$
 2 (enunciado)

seções delgadas simétricas

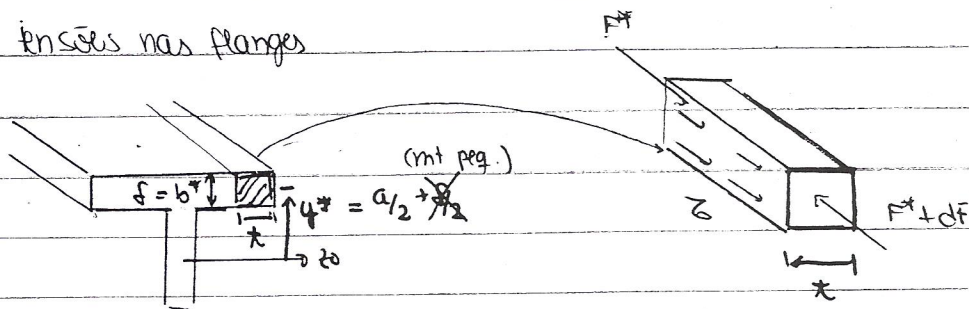


(pois $\delta \ll a$) desconsiderar

$$J_{z0} = \frac{\delta a^3}{12} + 2 \left[\frac{a \delta^3}{12} + (\delta a) \left(\frac{a}{2} \right)^2 \right]$$

$$J_{z0} = \frac{\delta a^3}{12} + \frac{\delta a^3}{2} = \frac{7 \delta a^3}{12}$$

tensões nas flanges



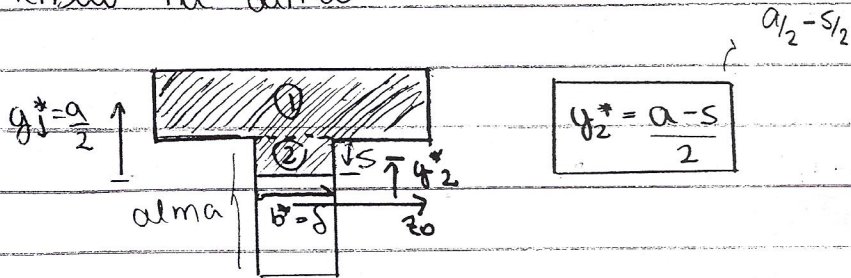
$$1) \quad M_s^* = S^* y^* = (\delta a) \frac{a}{2} = \frac{\delta a^2}{2}$$

$$2) \quad \bar{\sigma} = \frac{VM^*}{b^* J_{z0}} = \frac{V \cdot \frac{\delta a^2}{2}}{\delta \cdot \frac{7}{12} \delta a^3} =$$

$$\left. \begin{aligned} \bar{\sigma}(x=0) &= 0 \\ \bar{\sigma}(x=a/2) &= \frac{3}{7} \frac{V}{\delta a} \end{aligned} \right\} \bar{\sigma} = \frac{6}{7} \frac{V}{\delta a^2} \cdot x \quad \leftarrow = \frac{6}{7} \frac{V \delta a}{\delta^2 a^3}$$

porta vertical

Tensão na alma



$$M_s^* = M_s^{(1)} + M_s^{(2)} = \delta a \frac{a}{2} + (\delta \cdot s) \left(\frac{a-s}{2} \right) = \frac{\delta}{2} (a^2 + sa - s^2)$$

$$\tau = \frac{V M_s^*}{b^* I_{z0}} = \frac{6V}{7} \frac{(a^2 + sa - s^2)}{\delta a^3}$$

$$\tau(s=0) = 6/7 \cdot V/\delta a$$

$$\tau(s=s/2) = 15/14 \cdot V/\delta a = \tau_{max}$$

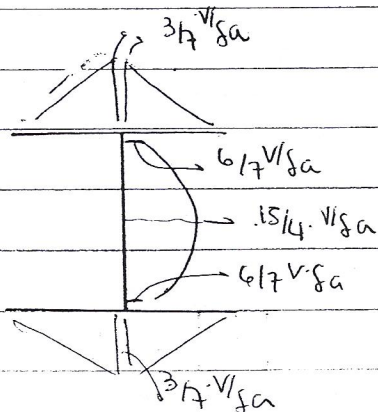
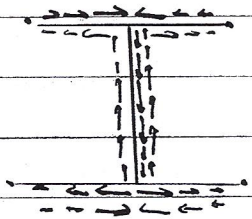
$$\tau(s=a) = 6/7 \cdot V/\delta a$$

fluxo de cisalhamento

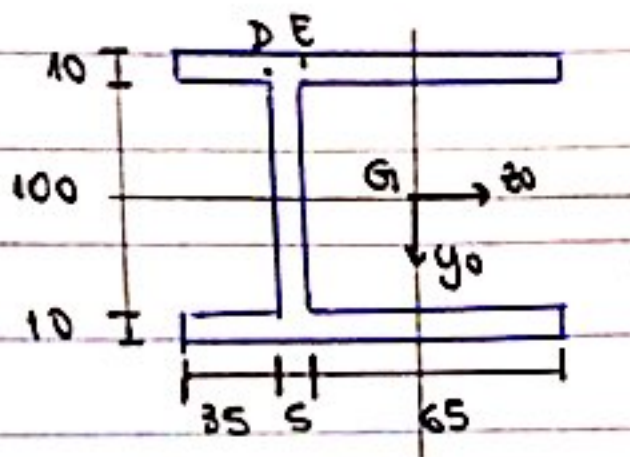
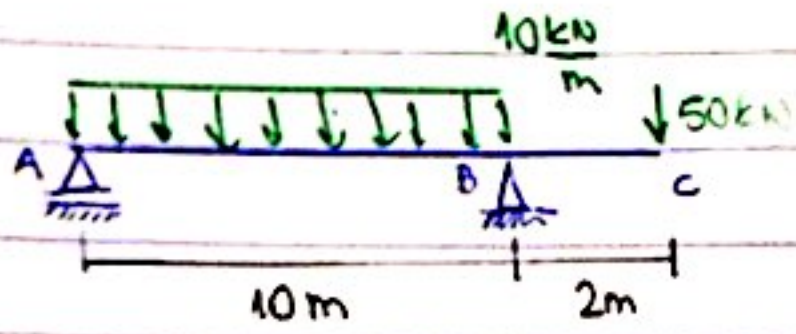
(valor da) constante

∴ nas pontas é sempre zero

se V ↑

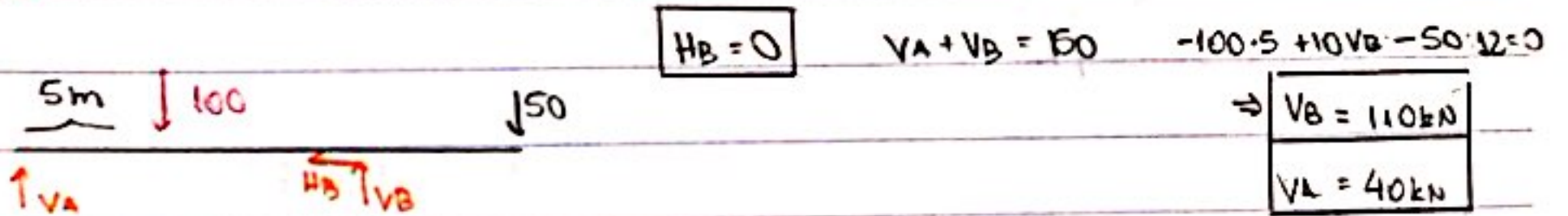


Exercício Determine o fluxo de cisalhamento para as regiões de maior esforço cortante e a região em D e E.

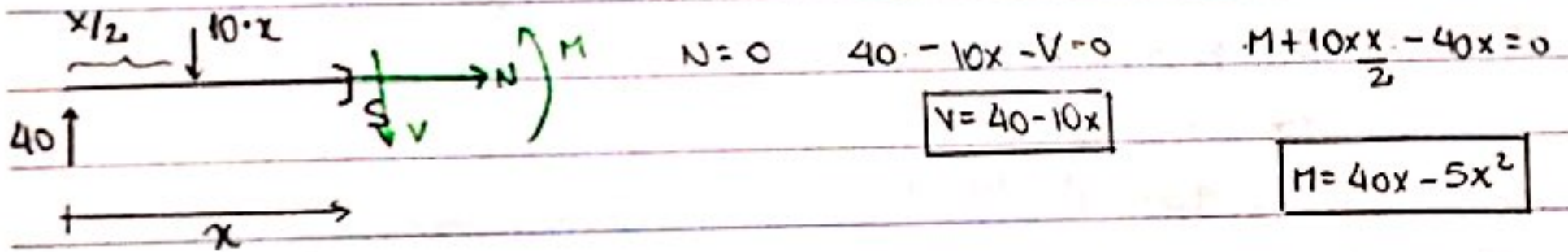


$I_{z0} = 6483 \cdot 10^3 \text{ m}^4$

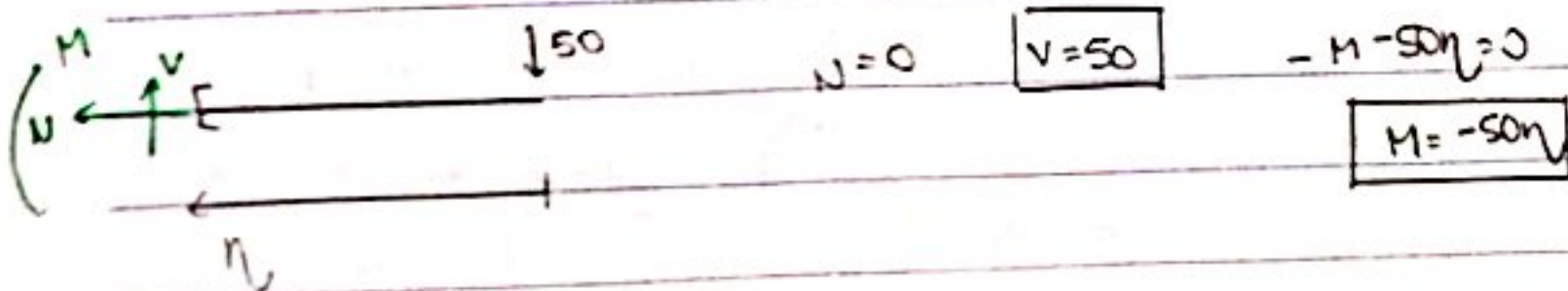
DCL



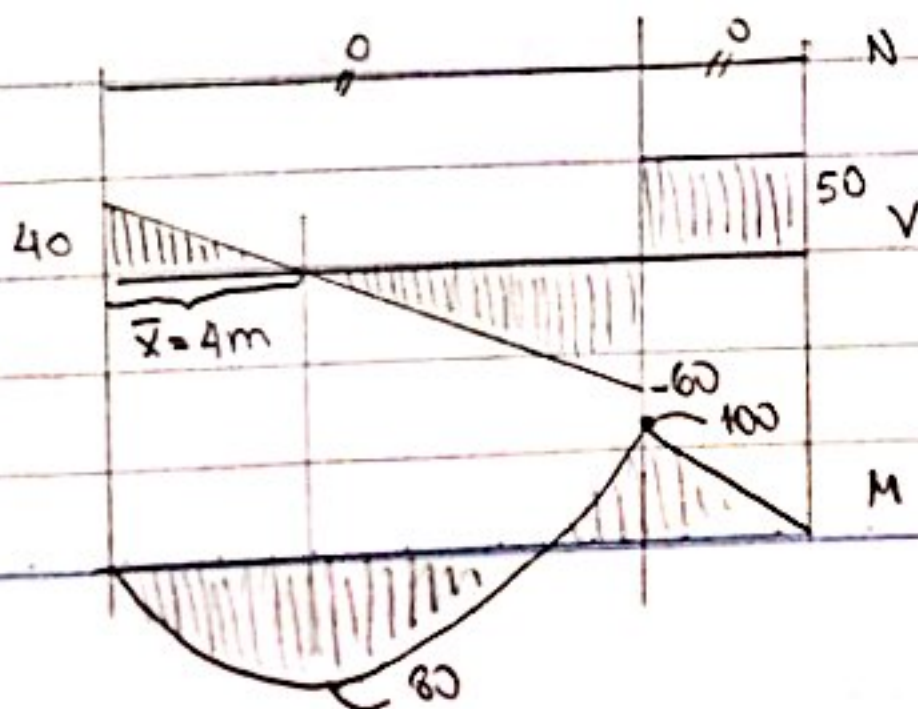
Trecho AB



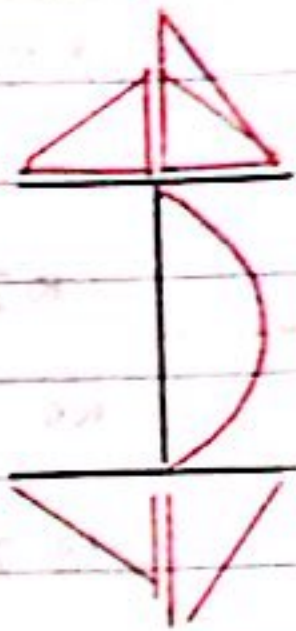
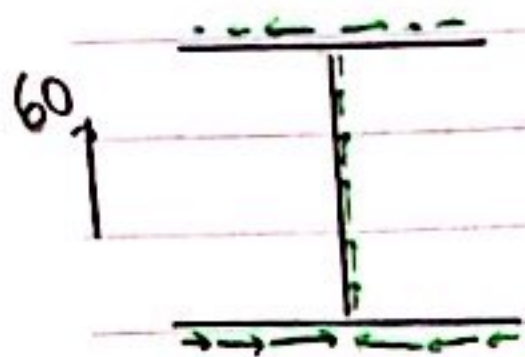
Trecho BC



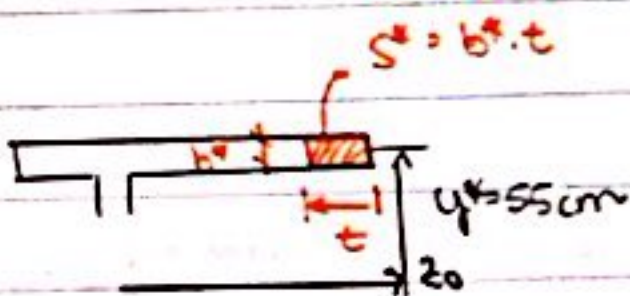
Diagramas



Maiores $V = -60 \text{ kN}$ //



(a barra \bar{n} é simétrica,
por isso o diagrama
é maior do lado direito)



$b^* = 10 \text{ cm}$ $S = 10 \text{ t}$ $M^* = S y^* = 550 \text{ t}^2$

$$\tau = \frac{V M^*}{b^* J_{20}} = \frac{60 \times 10^3 \cdot 550 \text{ t} \cdot 10^6}{91.6473 \text{ t}^4} \quad [\text{N/m}^2]$$

$\tau(x=35) = \tau_D = 15,27 \text{ MPa}$

$\tau(x=65) = \tau_B = 33,09 \text{ MPa}$

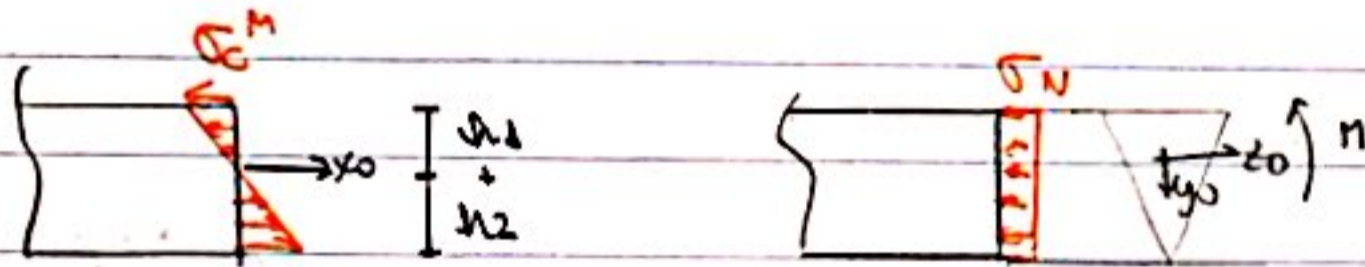
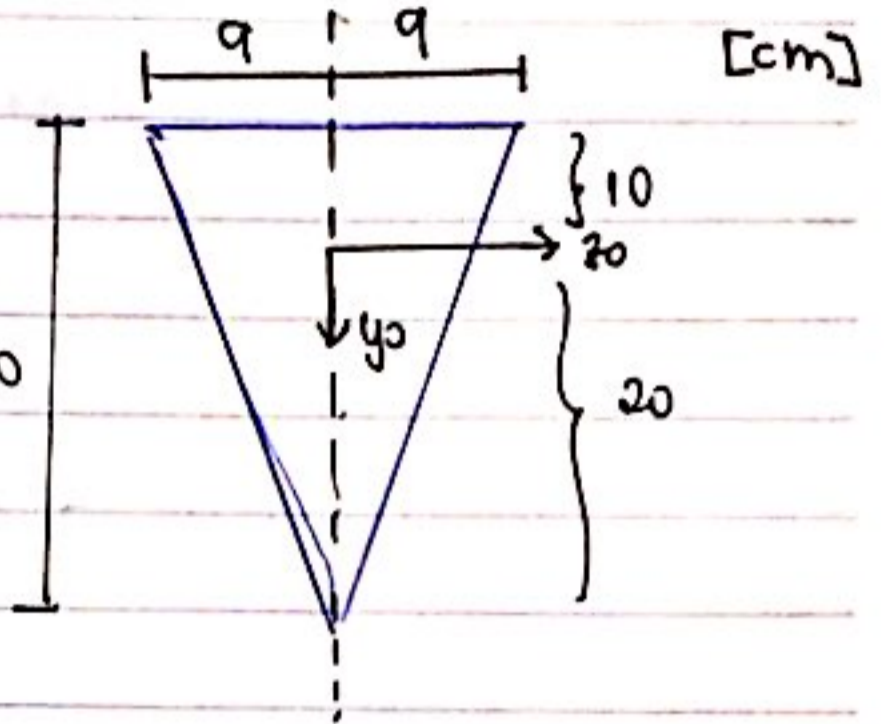
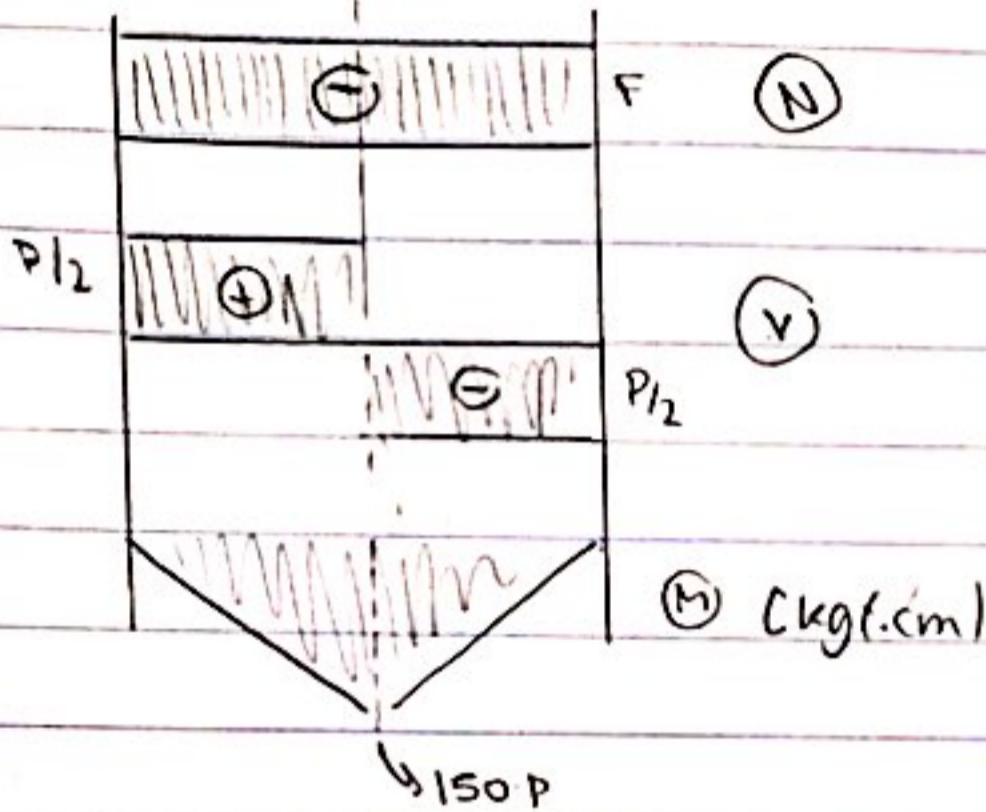
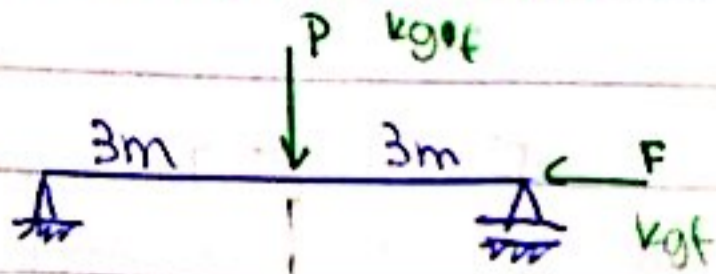
$\frac{V}{J_{20}} = \frac{60 \cdot 10^3}{91.6473} = 9,255 \times 10^3 \text{ N/mm}^4$ (corrigir unidades)

Revisão:

- * Figuras planas (2 matrículas)
- * Flexão pura $\sigma = M / J_{20} \cdot y$ $N=0$ $V=0$ $M=Cte$
- * " simples $\sigma(x,y) = \frac{M(x)}{J_{20}(x)} \cdot y$ $N=0$, $V \neq 0$ $M \neq Cte$
- * " composta $\sigma = \frac{M}{A} + \frac{M y}{J_{20}}$ $N \neq 0$, $V \neq 0$, $M \neq Cte$
- * Cisalhamento na flexão: $\tau = \frac{V M^*}{b^* J_{20}}$

2) Ex. da apostila (medida)

- a) Qual o máximo valor de P? e quando P assumir em valor, quanto vale F?
 b) Quando F=0, qual o máximo P?



a normal é de compressão

$$\left\{ \begin{aligned} \sigma_T^{\max} &= -\frac{F}{A} + \frac{M \cdot h_2}{I_{20}} \\ \sigma_C^{\max} &= -\frac{F}{A} + \frac{M \cdot (-h_1)}{I_{20}} \end{aligned} \right.$$

Dados:
 $\bar{\sigma}_T = 800 \text{ kgf/cm}^2$
 $\bar{\sigma}_C = 1100 \text{ kgf/cm}^2$

(1) $A = \frac{18 \cdot 30}{2} = 270 \text{ cm}^2$

(2) $I_{20} = \frac{18 \cdot 30^3}{30} = 13500 \text{ cm}^4$

Logo, (*) $\sigma_T^{\max} = \frac{-F}{270} + \frac{150P \cdot 20}{13500} \leq \bar{\sigma}_T$ $\frac{-F}{270} + \frac{2P}{9} \leq 800$

$|\sigma_C^{\max}| \leq \bar{\sigma}_C$ $\frac{F}{270} + \frac{150P \cdot 10}{13500} \leq 1100$ $\frac{F}{270} + \frac{P}{9} \leq 1100$

No limite: $\begin{cases} -F + 60P = 216000 \\ F + 30P = 297000 \end{cases} \Rightarrow \begin{cases} P_{\max} = 5700 \text{ kgf} \\ F = 126000 \text{ kgf} \end{cases}$

Rna $F=0$

$$\frac{P}{9} \leq 1100 \Rightarrow P \leq 9900 \text{ kgf}$$

$$\frac{2P}{9} \leq 800 \Rightarrow P \leq 3600 \text{ kgf}$$

$$\left. \begin{array}{l} \\ \\ \end{array} \right\} \Rightarrow \text{Logo, } \boxed{P_{\text{max}} = 3600 \text{ kgf}}$$